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THE McSHANE INTEGRAL OF BANACH-VALUED FUNCTIONS

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The generalized Riemann integral has, as the name suggests, a definition similar to the Riemann integral. The difference lies in the class of partitions that are used to form the Riemann sums. Two such generalizations have been studied for real-valued functions. One of these generalizations leads to an integral, often called the Henstock integral, that is equivalent to the restricted Denjoy integral while the other yields an integral, which we will refer to as the McShane integral, that is equivalent to the Lebesgue integral. We shall confine our attention to the latter definition and develop the properties of this integral for the case in which the function has values in a Banach space. The main result of this paper is that every measurable, Pettis integrable function is generalized Riemann integrable.

Throughout this paper X will denote a real Banach space and X^* its dual. We first extend the notion of partition of an interval.

DEFINITION 1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A tagged interval $(s, [c, d])$ consists of an interval $[c, d] \subset [a, b]$ and a point s in $[a, b]$. The tagged interval $(s, [c, d])$ is subordinate to δ if

$$[c, d] \subset (s - \delta(s), s + \delta(s)).$$

Note that this may not be a point in $[c, d]$. Script capital letters such as $\mathscr{P}$ and $\mathscr{D}$ will be used to denote finite collections of non-overlapping tagged intervals. Let

$$\mathscr{P} = \{(s_i, [c_i, d_i]) : 1 \le i \le N\}$$

be such a collection in $[a, b]$.

(a) The points $\{s_i: 1 \le i \le N\}$ are called the tags of $\mathscr{P}$.

(b) The intervals $\{[c_i, d_i]: 1 \le i \le N\}$ are called the intervals of $\mathscr{P}$.

(c) If $(s_i, [c_i, d_i])$ is subordinate to δ for each i, then we write $\mathscr{P}$ is sub δ.

(d) If $[a, b] = \bigcup_{i=1}^{N} [c_i, d_i]$, then $\mathscr{P}$ is called a tagged partition of $[a, b]$.

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