ILLINOIS JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 1991

THE FRÖLICHER SPECTRAL SEQUENCE FOR COMPACT NILMANIFOLDS

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1. Introduction

In 1955 A. Frölicher [Fr] defined a spectral sequence $\{E_r\}$ for any complex manifold M. We call it the *Frölicher spectral sequence*, but it is sometimes known as the Hodge-deRham spectral sequence. Nowadays the construction of this spectral sequence is standard, once one notes that the complex differential forms on M form a differential graded **C**-module.

For a compact manifold with positive definite Kähler metric Frölicher observed that $E_1 \cong E_\infty$; Kodaira [Kod] proved that the same conclusion holds for any compact complex surface. The Iwasawa manifold I_3 (defined as the quotient of the complex Heisenberg group by the Gaussian integers) has a nonclosed holomorphic 1-form, and so $E_1^{1,0}(I_3) \ncong E_2^{1,0}(I_3)$. Nevertheless $E_2(I_3) \cong E_\infty(I_3)$; more generally it follows from a result of Sakane (see Theorem 9 for a proof) that $E_2(M)) \cong E_\infty(M)$ for a compact complex parallelizable nilmanifold M. (A well-known result of Wang [Wa] asserts that any compact complex parallelizable manifold is the quotient of a complex Lie group by a discrete subgroup.)

In spite of the fact that Frölicher's paper has been in existence for more than 30 years, until recently no examples of complex manifolds for which $E_2 \ncong E_\infty$ seem to have been known (see [GH, page 444]). In our note [CFG] we found compact complex manifolds of complex dimension at least 4 for which $E_2 \ncong E_3 \cong E_\infty$. Since our examples are compact nilmanifolds, they are never simply connected. H. Pittie [Pi] has found some compact simply connected examples, the simplest of which is *Spin*(9). All of Pittie's complex manifolds must have much larger dimensions than ours.

The principal fact that led us to our examples is the observation that there are many compact nilmanifolds which possess complex structures but are not complex parallelizable. Such a manifold M is real parallelizable, however, and moreover both the deRham operator d and the Dolbeault operator $\bar{\partial}$ have explicit descriptions in terms of a canonical parallelization. Although the calculations become complicated when the dimension of M is large, it is

Received October 12, 1988.