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ON HYPERSURFACES OF LIE GROUPS

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0. *Introduction.* In his work on the theory of surfaces Gauss introduced what is called today the (normal) Gauss map of an orientable hypersurface in Euclidean space E^n.

Considering E^n as a commutative Lie group, the Gauss map is just the translation of the normal vector of the hypersurfaces at any point to the identity. Reasoning this way, we can also consider such a map in an orientable hypersurface of an arbitrary Lie group. In this work, we use this map to obtain some results on the geometry and topology of a hypersurface in a Lie group which apply, in particular, to S^3 and E^n. To state the results, let us consider a Lie group G with a left invariant metric, an orientable hypersurface M of G, and a map $\gamma\colon M \to \Gamma$, Γ being the Lie algebra of G, which (left) translate the normal vector of M at any point to the identity of G.

As in Euclidean spaces, the tangent space of M at any point can be identified, up to a left translation, with the tangent space of the unit sphere of the Lie algebra of G. Therefore, the derivative of γ can be considered a tensor on TM. Its determinant, say K, in E^n coincides with the Gauss-Kronecker curvature of M (that is, the determinant of the 2nd fundamental form of M), which is the intrinsic curvature of M when $n = 3$. In S^3, we prove that K is also the intrinsic curvature of M (§5). In general, the derivative of γ is the sum of the shape operator of M plus a tensor, depending essentially on G, which we call here the invariant shape operator of M (Definition 2 and Proposition 3). We will say that a point $x \in M$ is degenerate if $d\gamma(x) \equiv 0$.

We prove that *if M is compact and has no degenerate points, then either M is homeomorphic to a sphere or K is* zero *in a non denumerable subset of M* (Theorem 7).

When G is commutative, we have that x is degenerate if and only if x is totally geodesic. In general, if x is degenerate then the translation of $T_x(M)$ to the identity is a Lie subalgebra of codimension 1 (Proposition 10). It follows, in particular, that if G has finite center and the metric is bi-invariant, then M has no degenerate points.