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AUTOMATIC CONTINUITY IS EQUIVALENT TO UNIQUENESS OF INVARIANT MEANS

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0. Let G be a group and suppose that (X, B, m) is a probability space on which G acts as a group of invertible measure-preserving transformations. Given a function $f\colon X \to R$ and $g \in G$, the translation ${}_gf$ is defined by ${}_gf(x) = f(g^{-1}x)$ for all $x \in X$. This defines a representation of G as a group of isometries of the Lebesgue spaces $L_p(X)$, $1 \le p \le \infty$. This representation will be called the *regular representation of* G. A G*-invariant mean* (for this representation of G as measure-preserving transformation of (X, B, m)) is a positive linear functional ϕ on $L_\infty(X)$ such that $\phi(1) = 1$ and $\phi({}_gf) = \phi(f)$ for all $g \in G$ and $f \in L_\infty(X)$. The integral with respect to m, denoted $\int dm$, is one such G-invariant mean. We say that the action of G has a *unique* G*-invariant mean* if and only if $\int dm$ is the only G-invariant mean. Because there will not be a unique G-invariant mean when the group action by G is not ergodic, we may assume when discussing invariant means that G acts ergodically on (X, B, m); i.e., if $A \in B$ and $m(gA \,\Delta\, A) = 0$ for all $g \in G$, then either $m(A) = 0$ or $m(A) = 1$. The question of uniqueness of G-invariant means is of considerable interest and has been successfully settled in many cases in recent years. See Drinfeld [4], Margulis [7], Rosenblatt [12], Sullivan [15].

A G*-invariant linear form* on $L_p(X)$ is a linear functional λ on $L_p(X)$ such that $\lambda({}_gf) = \lambda(f)$ for all $g \in G$ and $f \in L_p(X)$. We say that the representation of G on (X, B, m) has L_p *automatic continuity* if any G-invariant linear form on $L_p(X)$ is continuous in the L_p-norm topology. If the σ-algebra of invariant sets is infinite, then generally L_p automatic continuity fails to hold, so we may assume that G acts ergodically on (X, B, m) when discussing automatic continuity. Notice also that if the action by G is ergodic, then any continuous G-invariant linear form on $L_p(X)$, $1 \le p < \infty$, must be a constant multiple of $\int dm$. This is not necessarily the case on $L_\infty(X)$ because of the possible non-uniqueness of G-invariant means. Whether or not the representation of G has automatic continuity can hinge on the algebraic and/or topological properties of G, as well as the function space

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