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CLOSED-FORM SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS VIA QUASI-SOLUTIONS II

BY

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Dedicated to my teacher, R. Creighton Buck, on the occasion of his retirement

Introduction

In this paper, we continue the work of Part I by studying a kind of separation of variables for some of the PDE's of mathematical physics. For example, in Section 1, we find all solutions of the form $u = \varphi(A(x) + B(y))$ of Laplace's equation in two variables, $u_{xx} + u_{yy} = 0$. In the terminology of Part I, we study which $A(x) + B(y)$ are quasi-solutions of Laplace's equation. (However, Part II can be read independently of Part I.) Surprisingly, the Jacobi elliptic functions appear naturally in this context. At present, their appearance seems to be an "accident" of computation. We have no conceptual explanation for their occurrence, or for why doubly-periodic functions should play a role in this kind of separation of variables.

Questions to be addressed at a future time involve expansions in series whose terms are constants times these Jacobi elliptic functions. How does one choose these constants? Are there any orthogonality relations? How fast is the convergence? These are natural questions especially for solving the Dirichlet problem in a rectangle, for which this form of separation of variables seems quite appropriate. Note that by using logarithms or exponentials, it is equivalent to the form $u = \psi(C(x)D(y))$.

Willard Miller, Jr., has shown the author how to derive many of the results of this paper by the method of differential-Stäckel matrices in [KAM]. This organized method has some advantages over the ad hoc methods of the present paper, but the calculations are still lengthy. The author thanks Professor Miller for his helpful communications.

Throughout this paper, all functions are supposed to be real-analytic on a domain in the appropriate Euclidean space. Alternatively, one could suppose

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