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LOGARITHMIC SOBOLEV INEQUALITIES ON LIE GROUPS

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1. Introduction

The heat kernel on a manifold provides a natural analog for n dimensional Gauss measure. If Δ is the Laplace-Beltrami operator on a Riemannian manifold then the heat kernel $p_t(x, \cdot)$ is the measure on M given by $(e^{t\Delta}f)(x) = \int_M f(y)p_t(x, dy)$. $p_t(x, \cdot)$ reduces to Gauss measure centered at x in case $M = R^n$. Put $t = 1$, fix x and write $\mu(dy) = p_1(x, dy)$. The Dirichlet form operator L for μ is the self-adjoint operator on $L^2(\mu)$ defined by

$$(1.1) \qquad (Lf, g)_{L^2(\mu)} = \int_M \operatorname{grad} f(y) \cdot \operatorname{grad} \bar{g}(y)\mu(dy)$$

for f and g in $C_c^\infty(M)$. In case $M = R^n$, L is the harmonic oscillator Hamiltonian in its ground state representation. Ever since E. Nelson [32] showed the usefulness to quantum field theory of operator bounds on $\exp(-sL)$ as an operator from $L^p(\mu)$ to $L^q(\mu)$, the boundedness and in particular the contractivity of this operator and similar operators has been explored with great intensity. A variety of techniques for exploring $\exp(-sL)$ as an operator from L^p to L^q for the harmonic oscillator Hamiltonian and for other second order elliptic operators have been investigated. Among them is the use of an equivalence between boundedness properties of e^{-tL}: $L^p \to L^q$ (hypercontractivity) and direct inequalities on the quadratic form of L itself. The latter have the form of logarithmic Sobolev inequalities [17] (see e.g. (3.7) below). A survey of these topics is given in [9] and a more recent bibliography is given in [18]. In the present paper it will be shown that a technique used in [17] for proving logarithmic Sobolev inequalities for Gauss measure on R^n goes over to the heat kernel measure on Lie groups.

Denote by W the space $C_0([0, T]; G)$ of continuous functions $g(\cdot)$ on $[0, T]$ with values in a connected Lie group G for which $g(0) =$ identity. In Section

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