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EXAMPLES OF SOLVMANIFOLDS WITHOUT CERTAIN AFFINE STRUCTURE[1]

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Let G be a connected and simply connected solvable Lie group. J. Milnor [Milnor] in 1977 conjectured that G admits a complete affinely flat structure invariant under left translations. Recently Boyom [Boyom] has published a positive answer for nilpotent groups using complicated Lie algebra and left symmetric algebra constructions. The general question remains open. See the note added in proof at the end of the paper.

Suppose G admits a lattice π. Let N be the nil-radical of G and $\Gamma = N \cap \pi$. Then Γ is a lattice in N. G is diffeomorphic to $\mathbf{R}^m$ and it is known that $\pi \backslash \mathbf{R}^m$ is a bundle over a torus with a nil-manifold $\Gamma \backslash N$ a fiber. Now, if there is an embedding of π into $\mathrm{Aff}(m) = \mathbf{R}^m \rtimes GL(m, \mathbf{R})$, $m = \mathrm{rank}(\pi)$ so that π acts as a group of affine diffeomorphisms of $\mathbf{R}^m$, it is tempting to expect that the center of Γ is carried into pure translations on $\mathbf{R}^m$ and that π/Γ also induces pure translations on $\mathbf{R}^{m-\mathrm{rank}(\Gamma)}$. In fact, one can expect that π imbeds into $\mathrm{Aff}(m) = \mathbf{R}^m \rtimes GL(m, \mathbf{R})$ in the blocked form

$$\left(\begin{bmatrix} A & M \\ O & I \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \mathbf{R}^m \rtimes GL(m, \mathbf{R})$$

where $A \in GL(k, \mathbf{R})$, $k = \mathrm{rank}(\Gamma)$ and O is the zero matrix; with the center of Γ mapping into pure translations.

This approach has proved successful for 2-step nilpotent Lie groups; see [Lee]. Many others, including [Nisse], have attempted to prove the conjecture using induction based on exploiting this idea. This technique, as we shall show, will, unfortunately, not work in general. We present here an example of a solvmanifold which is a torus bundle over a torus. We show that *it does not admit an affine structure of the expected type*. In particular, the expected type of embedding of this example would be "canonical" in the sense of Nisse. Therefore, our example shows that Proposition 2.1 and Théoreme B of [Nisse] is incorrect.

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