THE UPPER CENTRAL SERIES IN SOLUBLE GROUPS

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1. Introduction

1.1. This paper has grown from an attempt to study relations between the periodic structure of a soluble group and its commutator structure. The discussion will centre primarily around the upper central series and the four Engel radicals that we introduced in an earlier paper in this journal. (This paper is listed as [EE] in the references at the end.) We recall the definition of these radicals. If x is an element in a group G such that $\mathrm{Gp}\{x\}$, the subgroup generated by x, can be linked to G by a series (i.e., a well-ordered ascending normal system), then x is called a serial element in G (we write $x \infty\triangleleft G$), and the set of all such elements forms a subgroup $\sigma(G)$, whatever the nature of G. If $\mathrm{Gp}\{x\}$ can be linked to G by a finite series, x is called finitely-serial (we write $x \triangleleft\triangleleft G$), and the set of these also forms a subgroup, $\bar{\sigma}(G)$. Further, if $\rho(G)$ denotes the set of all $a \in G$ such that $x \infty\triangleleft a_x$ for every x, where a_x is the subgroup generated by x and all conjugates of a, then $\rho(G)$ is a subgroup of G; while if $\bar{\rho}(G)$ is the set of all a with the property that $x \triangleleft\triangleleft a_x$ for every x, and where the length of the series can be taken independent of x, then $\bar{\rho}(G)$ also is a subgroup. These four characteristic subgroups satisfy the inclusion relations

$$\alpha(G) \leqq \rho(G) \leqq \sigma(G) \leqq \eta(G);$$
$$\alpha_\omega(G) \leqq \bar{\rho}(G) \leqq \bar{\sigma}(G),$$

where $\alpha(G)$ and $\alpha_\omega(G)$, are the final and ω^{th} terms, respectively, of the upper central series of G, and $\eta(G)$ is the unique maximal locally nilpotent normal subgroup of G. (The group $\eta(G)$ was denoted by $\varphi(G)$ in [EE] and called there the Fitting radical of G. However, it seems preferable to reserve the symbol φ for the Frattini subgroup, and the term Fitting radical for the union in G of all nilpotent normal subgroups. The group $\eta(G)$ might be called the *Hirsch-Plotkin radical*, after Hirsch and Plotkin who discovered its existence. It should be remarked that, with the exception of the switch from $\varphi(G)$ to $\eta(G)$, all the notation and terminology introduced in [EE] will continue to be used in the present paper.)

It was shown in [EE] that there exist countably generated metabelian groups in which no two of the six subgroups determined by σ, $\bar{\sigma}$, ρ, $\bar{\rho}$, α, α_ω coincide. Our example had the form $U \times V \times W$, where U and W were periodic (in fact, finite and of exponent 4, respectively), but V was not: V was an extension of an abelian group A of type 2^∞ by an infinite cyclic group with generator b, where $b^{-1}ab = a^3$ for $a \in A$. If we let V_1 be the split extension of A by a

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