SOME REMARKS ON TABOO PROBABILITIES

BY

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Consider a discrete-parameter homogeneous Markov chain $\{x_n, n \geqq 0\}$ with state space $\mathbf{I}$ and one-step transition matrix $((p_{ij}))$, $i, j \in \mathbf{I}$. For any subset H of $\mathbf{I}$ we define the taboo probability

$$ {}_Hp_{ij}^{(n)} = \mathbf{P}\{x_n = j; x_\nu \notin H, 0 < \nu < n \mid x_0 = i\}, \qquad n \geqq 1, $$

and set

$$ {}_Hp_{ij}^{*} = \sum_{n=1}^{\infty} {}_Hp_{ij}^{(n)}. $$

When $H = \{k\}$ we write ${}_kp_{ij}^{(n)}$ for ${}_Hp_{ij}^{(n)}$. Furthermore we write $f_{ij}^{(n)}$ for ${}_jp_{ij}^{(n)}$ and $e_{ij}^{(n)}$ for ${}_ip_{ij}^{(n)}$. Thus,

$$ f_{ij}^{*} = \sum_{n=1}^{\infty} f_{ij}^{(n)}, \qquad e_{ij}^{*} = \sum_{n=1}^{\infty} e_{ij}^{(n)}. $$

The quantity f_{ij}^{*} is familiar, the quantity e_{ij}^{*} has been studied in [1] under the notation e_{ij}. We set also

$$ m_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)}; $$

and ${}_Hp_{ij}^{(0)} = \delta_{ij}$.

In this paper a "well known" statement means one which can be found in [1], particularly §I.9 there which treats taboo probabilities. What follows may be regarded as some interesting corollaries of well-known results which seem worth stating. They are engendered by a generalization (Proposition 7) of a recent result of Spitzer [3]. This will be placed where it properly belongs, and the proof will be strictly elementary. In doing so we shall define a new binary relation between the states of a Markov chain.

Recall that two states i and j belong to the same recurrent class if and only if $f_{ij}^{*} = f_{ji}^{*} = 1$, or alternatively $f_{ii}^{*} = 1$ and $f_{ij}^{*} > 0$. A subset C of $\mathbf{I}$ is said to form an *equitable class* if and only if for every i and j in C we have

$$ (1) \qquad e_{ij}^{*} = 1. $$

A well-known example of an equitable class is the following: x_n is the sum of n independent and identically distributed random variables with mean zero, or more generally, x_n is a recurrent Markov chain with stationary and independent increments.

We have the following characterization.

Proposition 1. *An equitable class C is recurrent. A recurrent class is equitable if and only if for each i in C, we have*

$$ (2) \qquad \sum_{j \in C} p_{ji} = 1, $$

namely when $((p_{ij}))$ restricted to C is doubly stochastic.

Received October 6, 1960.

[1] This work is supported in part by the Office of Scientific Research of the United States Air Force.