ON THE RELATION BETWEEN THE DETERMINANT AND THE PERMANENT[1]

BY

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In this paper we consider the problem of determining whether or not there exist linear operations on matrices that change their permanents into their determinants. Recent interest in the permanent function stems from its application to certain combinatorial problems [4; p. 166] and from an unresolved conjecture of van der Waerden [2].

If X is an n-square matrix, then the permanent of X is defined by

$$\operatorname{per}(X) = \sum_\sigma \prod_{i=1}^n x_{i\sigma(i)} ,$$

where σ runs over all permutations of $1, \cdots, n$. We introduce a notation to simplify writing sets of indices: if $1 \leqq r \leqq n$, then $Q_{n,r}$ will denote the totality of increasing sequences $\omega: 1 \leqq i_1 < \cdots < i_r \leqq n$. If X is an $m \times n$ matrix and $\omega \in Q_{m,r}$, $\tau \in Q_{n,r}$, then $X^+_{\omega,\tau}$ will denote the permanent of the submatrix of X with row indices ω and column indices τ. The symbol $X_{\omega,\tau}$ will denote the determinant of this submatrix. If s is an index in ω, then ω'_s will denote the sequence ω with s omitted. In case $\omega \in Q_{m,r}$, $\tau \in Q_{n,r}$, we will reserve the special notation $E_{\omega,\tau}$ for the $\binom{m}{r} \times \binom{n}{r}$ *unit matrix* with 1 in the (ω, τ) position in the doubly lexicographic ordering, zero elsewhere. That is, we imagine the rows of $E_{\omega,\tau}$ indexed with the elements in $Q_{m,r}$ where the ordering is the lexicographic one, and similarly for the columns. If $u_1, \cdots, u_p$ are vectors in some space V, we denote by $\langle u_1, \cdots, u_p \rangle$ the subspace of V spanned by these vectors. Finally, $\rho(X)$ will denote the rank of the matrix X; $[u, v] \parallel [x, y]$ will mean that the two vectors are linearly dependent, and $[u, v] \perp [x, y]$ that they are orthogonal, i.e., that $ux + vy = 0$. Let $2 \leqq r \leqq \min(m, n)$, and denote by $M_{m,n}$ the vector space of $m \times n$ matrices over a field F of characteristic zero. To fix the notation assume henceforth that $m \leqq n$. Then $C_r(X)$ and $P_r(X)$ will denote the r^{th} *determinantal* and *permanental compound matrices* of X, respectively. That is, $C_r(X)$, $(P_r(X))$, is the matrix in $M_{\binom{m}{r},\binom{n}{r}}$ whose entries are the r-square subdeterminants (subpermanents) of X arranged in doubly lexicographic order. Let Δ be the map on $M_{2,2}$ into itself where $(\Delta(X))_{21} = -x_{21}$ and $(\Delta(X))_{ij} = x_{ij}$ otherwise. Note that $\operatorname{per}(X) = \det(\Delta(X))$.

Theorem. *There is no linear transformation T of $M_{m,n}$ into itself such that*

$$P_r(T(X)) = S_r\, C_r(X) \tag{1}$$

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