ON THE NUMBER OF NILPOTENT MATRICES WITH COEFFICIENTS IN A FINITE FIELD

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Fine and Herstein have demonstrated [1] that the number of nilpotent $n \times n$ matices with coefficients in the finite field of q elements, $GF(q)$, is q^{n^2-n}. The present (self-contained) note gives an alternate proof, suggested by algebraic geometry, and not involving sums over partitions of n. Using a lemma of [1], Reiner [2] has determined the number of matrices over $GF(q)$ having a given characteristic polynomial. This result is here obtained directly from the Fine-Herstein theorem.

1. Proof of the Fine-Herstein theorem

Throughout, N_k will denote the $k \times k$ matrix having zeros everywhere but on the first diagonal above the principal one and unity everywhere there. If $k = n$, we write simply N. Given a nilpotent $n \times n$ matrix A, we shall denote by $L(A)$ the linear space of all matrices Y such that $NY = YA$, and by $\mathfrak{A}$ the union of the spaces $L(A)$ for all nilpotent A. The matrices in $\mathfrak{A}$ will be called admissible. One sees that $L(T^{-1}AT) = L(A)T$, whence Y is admissible if and only if YT is admissible for any nonsingular T.

We determine now a necessary and sufficient condition that $Y \in \mathfrak{A}$. Given Y, let v be a row vector such that $vY = 0$. If $Y \in \mathfrak{A}$, then $vNY = vYA$ for some A, so $(vN)Y = 0$, i.e., the null space of Y is preserved by N. Let $v = (v_1, \cdots, v_n)$. Then $vN = (0, v_1, \cdots, v_{n-1})$. This implies that if the rank of Y is r, then the last $n - r$ rows of Y are zero (and the first r, therefore, are independent). Conversely, suppose Y has this property. Then for some nonsingular T, $YT = E$ is the direct sum of the $r \times r$ identity matrix I_r and the $(n - r) \times (n - r)$ zero matrix O_{n-r}. Now E is admissible, for $NE = N_r \oplus O_{n-r}$ is nilpotent and $NE = E(NE)$. Therefore $Y = ET^{-1}$ is admissible. The necessary and sufficient condition that $Y \in \mathfrak{A}$ is therefore

If rank $Y = r$, then the last $n - r$ rows of Y vanish.

We see next that $\dim L(A) = n$ for all nilpotent A. Observe that $NY = YA$ implies $N^kY = YA^k$ for all k. Let e_i denote the row vector having one in the i^{th} place and zeros elsewhere. Then $e_1 Y$ (i.e., the first row of Y) may be prescribed arbitrarily, but $e_k Y$ is then determined for all k by the relation $e_k Y = e_1 N^{k-1}Y = e_1 YA^{k-1}$.

Given $Y \in \mathfrak{A}$, let there be assigned to it a multiplicity $m(Y)$ equal to the number of distinct nilpotent matrices A such that $NY = YA$. If O is the zero matrix, then $m(O)$ is just the number of all nilpotent matrices, which we

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