THE DERIVED SERIES OF A FINITE p-GROUP[1]

BY

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The Galois groups of a class field tower form a chain of finite groups $G_1, G_2, \cdots$, such that G_1 is abelian and $G_n \cong G_{n+1}/G_{n+1}^{(n)}$, where $G_{n+1}^{(n)}$ denotes the n^{th} derived group of G_{n+1}. The class field tower and the chain of groups terminate after n steps if $G_{n+1}^{(n)} = \langle 1 \rangle$. We shall consider the case where all G_n are p-groups. It is known [5] that the chain terminates if G_1 is cyclic, or if $p = 2$ and G_1 has type $(2, 2)$. Olga Taussky (see Magnus [4]) posed the problem of determining whether such a chain of p-groups must always terminate. N. Itô [3] gave a negative answer to this question by constructing an infinite chain of p-groups satisfying the above conditions with G_1 of type (p, p, p) and $p \neq 2$. The question of the existence or nonexistence of infinite chains with G_1 generated by two elements or with $p = 2$ remained open.

The main result of this paper is the following theorem.

THEOREM 1. *Suppose $p \neq 2$, and let G be a noncyclic abelian p-group. Then there exists an infinite chain of p-groups $G_1, G_2, \cdots$, such that*

$$G_1 \cong G, \quad G_n \cong G_{n+1}/G_{n+1}^{(n)}, \quad \text{and} \quad G_{n+1}^{(n)} \neq \langle 1 \rangle.$$

A weaker result is obtained if $p = 2$.

THEOREM 2. *Suppose G is an abelian 2-group which contains a subgroup having one of the types $(2^2, 2^3)$, $(2^2, 2^2, 2^2)$, $(2^2, 2^2, 2, 2)$, or $(2, 2, 2, 2, 2)$. Then there exists an infinite chain of 2-groups $G_1, G_2, \cdots$, such that $G_1 \cong G$, $G_n \cong G_{n+1}/G_{n+1}^{(n)}$, and $G_{n+1}^{(n)} \neq \langle 1 \rangle$.*

As we noted above, the chain $G_1, G_2, \cdots$ terminates if G_1 is cyclic, or if $p = 2$ and G_1 has type $(2, 2)$. The remaining cases not covered by Theorem 2 are undecided. The proof of Theorem 2 is similar to that of Theorem 1 and will not be given here. Full details can be found in the author's thesis [2].

A second question posed by Olga Taussky [6] can be stated as follows. Can a bound on the derived length of a p-group H be determined from the type of $H/H^{(1)}$? Such a bound exists if $H/H^{(1)}$ is cyclic or of type $(2, 2)$. W. Magnus [4] showed that there is no bound if $H/H^{(1)}$ has type $(3, 3, 3)$. A complete answer to this question for $p \neq 2$, and a partial answer for $p = 2$, is given by the next theorem.

THEOREM 3. *Suppose H is a p-group and $G = H/H^{(1)}$. The derived length*

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