COMPLETIONS OF GROUPS OF CLASS 2[1]

BY
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1. Introduction

Let G be a group with center Z and commutator subgroup C, and suppose that $G \supseteq Z \supseteq C$. If H is a complete ($nH = H$ for all $n > 0$) nilpotent group of class 2 that contains G and no proper complete subgroup of H contains G, then we say that H is a *completion* of G. We prove (Theorem 3.2) that there exists a completion of G if and only if $\{g \in G : ng \in C \text{ for some } n > 0\} \subseteq Z$. If C is torsion free, then there exists a completion of K of G such that the commutator subgroup of K is torsion free and the center of K is the abelian completion of Z. Moreover, any other such completion of G is isomorphic to K (Theorem 3.3). These results generalize the corresponding results of Baer for abelian groups, and also Vinogradov's result for torsion free G.

The author originally had a long transfinite proof of Theorem 2.1, and all other results were restricted by the hypothesis that G contains no elements of order 2. This hypothesis on G has been removed, and the author wishes to thank Reinhold Baer for suggesting the elegant proof of Theorem 2.1.

Notation. N and Δ will always denote additive abelian groups with elements $0, a, b, \cdots$ and $\theta, \alpha, \beta, \gamma, \cdots$ respectively. F will denote the group of all *factor mappings* of $\Delta \times \Delta$ into N. Thus $f \in F$ if and only if $f : \Delta \times \Delta \to N$ and for all $\alpha, \beta \in \Delta$

$$f(\alpha, \theta) = f(\theta, \beta) = 0,$$

and

$$f(\alpha, \beta + \gamma) + f(\beta, \gamma) = f(\alpha + \beta, \gamma) + f(\alpha, \beta).$$

Each $f \in F$ determines a central extension G of N by Δ, where $G = \Delta \times N$ and, for all (α, a) and (β, b) in G,

$$(\alpha, a) + (\beta, b) = (\alpha + \beta, f(\alpha, \beta) + a + b).$$

The mappings of $f, g \in F$ are *equivalent* if there exists $t : \Delta \to N$ such that for all $\alpha, \beta \in \Delta$,

$$f(\alpha, \beta) = g(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta).$$

In this case the mapping $(\alpha, a) \in G(\Delta, N, f)$ upon $(\alpha, a + t(\alpha))$ in $G(\Delta, N, g)$ is an isomorphism.

G will always denote an additive group with commutator group C and center Z, and we shall always assume that $G \supseteq Z \supseteq C$. Suppose that N is a subgroup of G between Z and C. Let $\Delta = G/N$, and let π be the natural homo-

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