SOME INEQUALITIES FOR POLYNOMIALS AND RELATED ENTIRE FUNCTIONS

BY

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1. Inequalities for polynomials

Throughout this section let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. The following results are immediate.

THEOREM A.

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqq n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{1}$$

THEOREM B. *For $R > 1$*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{2}$$

If $p(z)$ has no zeros in $|z| < 1$, Theorem A can be sharpened.

THEOREM C. *If $p(z)$ has no zeros in $|z| < 1$, then*

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqq \frac{n^2}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \tag{3}$$

Theorem C was proved by N. G. de Bruijn [4].

We prove a corresponding modification of Theorem B.

THEOREM 1. *If $p(z)$ has no zeros in $|z| < 1$, then*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \tag{4}$$

for $R > 1$.

Proof of Theorem 1. If $q(z) = z^n \overline{p(1/\bar{z})}$, then $|q(z)| = |p(z)|$ for $|z| = 1$. Since $p(z) \neq 0$ for $|z| < 1$, it follows that $|q(z)| \leqq |p(z)|$ for $|z| < 1$. On replacing z by $1/z$ we deduce that for $|z| > 1$,

$$|p(z)| \leqq |q(z)|.$$

Now $q(z) = \sum_{\nu=0}^{n} \bar{a}_{n-\nu} z^\nu$; hence

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leqq \frac{1}{2} \int_0^{2\pi} |q(Re^{i\theta})|^2 \, d\theta + \frac{1}{2} \int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta$$
$$= \pi \sum_{\nu=0}^{n} (R^{2\nu} + R^{2n-2\nu}) |a_\nu|^2.$$

Received December 16, 1959.