THE POWER SERIES COEFFICIENTS OF FUNCTIONS DEFINED BY DIRICHLET SERIES

BY

W. E. BRIGGS AND R. G. BUSCHMAN

If the Dirichlet series $f(s) = \sum_{n=1}^{\infty} h(n)n^{-s}$ has abscissa of convergence $\operatorname{Re} s = a$ and a simple pole at $s = a$, then $f(s)$ has the Laurent expansion

$$f(s) = \frac{C}{s-a} + \sum_{r=0}^{\infty} \frac{(-1)^r C_r}{r!} (s-a)^r.$$

The purpose of this paper is to derive expressions for the C_r and to list the results for various number-theoretic functions $h(n)$, thus generalizing the special case of $h(n) = 1$ found in [1]. It is assumed throughout that $f(s)$ is of the above form with C, a, $h(n)$, and C_r referring to this relation, and that $E(x) = \sum_{n \leqq x} h(n) - Ca^{-1}x^a = O(x^b)$ where $0 \leqq b < a$.

Two lemmas are stated without proof.

LEMMA 1. *If $b_1, b_2, b_3, \cdots$ is a sequence of complex numbers and $v(x)$ has a continuous derivative for $x > 1$, then*

$$\sum_{n \leqq x} b_n v(n) = (\sum_{n \leqq x} b_n) v(x) - \int_1^x (\sum_{n \leqq t} b_n) v'(t)\, dt.$$

LEMMA 2.

$$f(s) = s \int_1^{\infty} x^{-s-1} [\sum_{n \leqq x} h(n)]\, dx, \qquad \operatorname{Re} s > a$$

LEMMA 3. *If* $\operatorname{Re} s > b$, *then*

$$f_1(s) \equiv s \int_1^{\infty} x^{-s-1} E(x)\, dx = -\frac{C}{a} + \sum_{r=0}^{\infty} \frac{(-1)^r C_r}{r!} (s-a)^r.$$

Proof. The integral is an analytic function for $\operatorname{Re} s > b$ and equals $f(s) - C/a - C/(s-a)$ for $\operatorname{Re} s > a$.

THEOREM 1. *If $u < -b$, then*

$$\sum_{n \leqq x} n^u h(n) \log^r n = C \int_1^x t^{u+a-1} \log^r t\, dt + D_r + (-1)^r f_1^{(r)}(-u) + o(1),$$

where $D_r = C/a$ if $r = 0$ and $D_r = 0$ otherwise.

Proof. From Lemma 1

$$S = \sum_{n \leqq x} n^u h(n) \log^r n = \sum_{n \leqq x} h(n) x^u \log^r x - \int_1^x [\sum_{n \leqq t} h(n)] \frac{d}{dt} (t^u \log^r t)\, dt.$$

But $(d/dt)(t^u \log^r t) = (d^r/du^r)(ut^{u-1})$. Hence

$$S = Ca^{-1}x^{u+a} \log^r x + O(x^{u+b} \log^r x)$$
$$+ \frac{d^r}{du^r} \left\{ -u \int_1^{\infty} t^{u-1} E(t)\, dt - u \int_1^x Ca^{-1} t^{u+a-1}\, dt + u \int_x^{\infty} t^{u-1} E(t)\, dt \right\}.$$

Received July 8, 1959; received in revised form August 30, 1960.