ON UNIQUE FACTORIZATION DOMAINS

BY
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A unique factorization domain (or UFD) is an integral domain in which every element $\neq 0$ is, in an essentially unique way (i.e., up to units), a product of irreducible ones. In spite of the simplicity of this notion, many problems concerning it have remained open for many years. For example the fact that every regular local ring is a UFD has been conjectured in the early 40's, many partial results in this direction have been proved, but the general case has been settled only in 1959 by M. Auslander and D. Buchsbaum [1]. Another open question was as to whether a power series ring over a UFD is a UFD; W. Krull studied it in a paper of 1938, and termed the answer "doubtful" ("zweifelhaft") [3]; we prove here that the answer to this question is negative. However, using the result of Auslander-Buchsbaum, we prove that a power series ring in any number of variables over a PID ("principal ideal domain") is a UFD. We also show, by counterexamples, that unique factorization is preserved neither by ground-field extension, nor by ground-field restriction.

I have received great help and stimulation from my friends M. Auslander, I. Kaplansky, and especially D. Buchsbaum. More particularly, Lemma 3.3 is essentially due to D. Buchsbaum, whereas the ideas leading to the proof of Theorem 2.1 came from discussions between him and me; after these discussions we arrived independently at a proof of this result.

1. Some preliminary results

In this paper all rings are assumed to be commutative and *noetherian* Let A be a noetherian domain; it is well known that the following conditions are equivalent:

(UF. 1) A is a UFD.
(UF. 2) Any two elements of A have a g.c.d.
(UF. 3) Any two elements of A have a l.c.m.
(UF. 4) The intersection of any two principal ideals of A is principal.
(UF. 5) Any irreducible element of A generates a prime ideal.
(UF. 6) Any prime ideal of height 1 of A is principal.

Furthermore, if A is a *local or semilocal* ring, these conditions are equivalent to:

(UF. 7) For any two elements a, b of A, we have $\mathrm{dh}(Aa + Ab) \leqq 1$ (where dh denotes the homological dimension of a module).

We say that an element a of a ring A is *prime* if the ideal Aa is prime; any prime element is irreducible; the converse is true in a UFD (by (UF. 5)).

The following lemmas are known, but we state and prove them for the reader's convenience:

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