QUASI-FROBENIUS RINGS AND GALOIS THEORY[1]

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1. Introduction

In one of his fundamental papers on Frobenius algebras, Nakayama proved that if M is a finitely generated free left module for a quasi-Frobenius ring R, then $\mathrm{Hom}_R(M, M)$ is also a quasi-Frobenius ring. It is not necessarily true, however, that M is also a free $\mathrm{Hom}_R(M, M)$-module. This lack of symmetry is removed in the main result of this paper, which states that if M is a faithful finitely generated projective left module for a quasi-Frobenius ring R, then $\mathrm{Hom}_R(M, M)$ is a quasi-Frobenius ring, and M is a projective $\mathrm{Hom}_R(M, M)$-module. An example is given to show that $\mathrm{Hom}_R(M, M)$ is not always quasi-Frobenius if M is not required to be a projective R-module.

In §3 the theorem is applied to obtain sufficient conditions on a group G of automorphisms of finite reduced order of a simple ring $\mathfrak{L}$ with minimum condition in order that the subring of fixed elements be a quasi-Frobenius ring. A formula is derived for the reduced order of G in terms of the height and index relative to $\mathfrak{L}$ of the indecomposable right ideal direct summands of the fixed ring $I(G)$ of G. These results constitute a first step towards a classification of the subrings of a simple ring with minimum condition which are the fixed rings under groups of automorphisms of the simple ring. A quasi-Frobenius ring seems to be a logical candidate for a subring of fixed elements because it has the property that the double centralizer of any faithful module coincides with the set of scalar multiplications by elements of the ring, a property which any ring which is to play a role in the Galois theory must possess. The main problem remains unsolved, namely to characterize those quasi-Frobenius subrings of a simple ring with minimum condition which are the subrings of fixed elements of groups of automorphisms. We have also made no attempt to solve these problems for rings without chain conditions.

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2. A theorem on the structure of the centralizer of a module

Let R be a ring with an identity element, and let R satisfy the minimum condition for left and right ideals. We shall be concerned with left and right R-modules on which the identity element of R is always assumed to act as identity operator. We shall use without further comment the result that any finitely generated left or right R-module satisfies both chain conditions for submodules.

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