CLASSES OF PERIODIC SEQUENCES[1]

BY

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1. Introduction

In certain psychological experiments connected with the learning of periodic sequences of symbols,[2] it is reasonable to identify two sequences if we can get one from the other by beginning at a different point, by permuting the symbols, or by a combination of these operations. For example, if there are two symbols, say 0 and 1, and if the period is 3, the following sequences are equivalent:

$$(011), \quad (101), \quad (110), \quad (100), \quad (010), \quad (001).$$

Also, (000) and (111) are equivalent. Thus, the eight possible sequences fall into two equivalence classes. It is of interest to determine how many classes there are for a given period n. Since a sequence of period n also has period kn, where k is any integer, there will be duplications as we run through all periods. For example, the class $\{(000), (111)\}$ will already have been counted for $n = 1$ and $n = 2$. We should therefore determine, for each n, the number of classes of sequences which have period n but no smaller period; that is, the number $F(n)$ of classes[3] with *primitive period* n. In our example, $F(1) = 1$, $F(2) = 1$, $F(3) = 1$, $F(4) = 2$, and so forth. If $F^*(n)$ denotes the total number of classes with period n, whether primitive or not, then

$$F^*(n) = \sum_{d \mid n} F(d), \tag{1}$$

where the summation is over all (positive) divisors d of n. This follows from the fact that every class of period n has a primitive period d which divides n.

2. Formulation

Let A denote the set of all periodic sequences

$$a = (\cdots, a_{-1}, a_0, a_1, \cdots),$$

where the a_j may take any of the q values $1, 2, \cdots, q$. Let Q denote the symmetric group on these q symbols, its elements being denoted generically by π, the identity element by e. Let T be the infinite cyclic group generated by the element τ. We can make Q and T act on A by the following rules:

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[2] E. H. GALANTER AND M. KOCHEN, *The acquisition and utilization of information in problem solving and thinking*, Lab Memo, University of Pennsylvania, 1957, and E. H. GALANTER AND W. A. S. SMITH, *Some experiments on thought*, Lab Memo, University of Pennsylvania, 1957.

[3] It is easily verified that equivalent sequences have the same periods, so it makes sense to speak of a class with period n.