ON A CLASS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Consider the system of linear differential equations

$$y'' + A(\lambda)y = \lambda\phi(t, \lambda)y + \lambda\psi(t, \lambda)y' \qquad (' = d/dt) \tag{1}$$

where λ is a real parameter, $y = (y_1, \cdots, y_n)$, $A(\lambda) = \text{diag}(\sigma_1^2, \cdots, \sigma_n^2)$, ϕ and ψ are $n \times n$ matrices whose elements are real, periodic functions of t of period $T = 2\pi/\omega$, are L-integrable in $[0, T]$, are analytic in λ and have mean value zero. Further, suppose that each $\sigma_j^2(\lambda)$, $j = 1, 2, \cdots, n$ is a real positive analytic function of λ with

$$\sigma_j(0) \not\equiv \sigma_h(0), \ (\text{mod } \omega i), \qquad j \neq h, \quad j, h = 1, 2, \cdots, n.$$

Systems of type (1) for $|\lambda|$ small have recently been extensively investigated by a method which has been successively developed and modified by L. Cesari, R. A. Gambill and J. K. Hale for both linear [1, 4, 5, 6, 9] and weakly nonlinear differential systems [7, 10]. The aim of the present paper is to prove a theorem, concerning the boundedness of the AC (absolutely continuous) solutions of (1), which contains as a particular case one of the various theorems proved in [1] and [4]. Applying the methods of [1], we prove the following:

THEOREM. *If*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix},$$

where ϕ_{ij}, ψ_{ij} are matrices with ϕ_{11} and ψ_{11} of dimension $\mu \times \mu$, and if (α) ϕ_{11}, ϕ_{22}, ψ_{21}, ψ_{12} are even in t, (β) ϕ_{21}, ϕ_{12}, ψ_{11}, ψ_{22} are odd in t, then, for $|\lambda|$ sufficiently small, all the AC *solutions of* (1) *are bounded in* $(-\infty, +\infty)$.

For ψ identically zero, ϕ and A independent of λ, and each element of ϕ an even function of t having mean value zero and possessing absolutely convergent Fourier series, this theorem was first proved by L. Cesari [1] and then extended by the author [9] to L-integrable functions. Using the techniques in [1], R. A. Gambill [4] extended the theorem of Cesari to the case where ψ is odd in t.

We shall prove the above theorem by showing that there is a fundamental system of AC solutions of (1) which are bounded for all values of t. Furthermore, we shall see that each solution y of the fundamental system so obtained has the following property: if the first μ components of y are even (or odd), then the last $n - \mu$ components are odd (or even).

If we make the transformation of variables

$$y_j = \frac{1}{2i\sigma_j}(z_{2j-1} - z_{2j}), \qquad y_j' = \tfrac{1}{2}(z_{2j-1} - z_{2j}), \qquad j = 1, 2, \cdots, n, \tag{2}$$

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