ON MODULES OF TRIVIAL COHOMOLOGY OVER A FINITE GROUP

BY TADASI NAKAYAMA

We say that a module A over a finite group G is *of trivial cohomology* when we have $H^n(H, A) = 0$ for every integer n and every subgroup H of G. Recently the writer proved:

THEOREM. *Let A be a module over a finite group G. Assume that for every prime p dividing the order $[G]$ of G there is an integer r_p such that*

$$H^{r_p}(H_p, A) = H^{r_p+1}(H_p, A) = 0,$$

where H_p is a p-Sylow subgroup of G. Then the G-module A is of trivial cohomology.

Our proof of this theorem in [8] (or [7]) was a combination of representation-theoretical arguments and an argument by so-called fundamental exact sequences in group cohomology. In the present note we shall give two (partly) new proofs, one by means of fundamental exact sequences only, like former proofs of a weaker form of the theorem ([5], [1], [2]), and one quite representation- or module-theoretical. Indeed, in the course of our latter proof, which makes use of an idea of Gaschütz, we shall obtain a result which may be considered as a structural characterization of a module of trivial cohomology.

1. Proof by fundamental exact sequences

Let G be a group and H an invariant subgroup of G. The theorem of fundamental exact sequences in group cohomology states [6], [4]: If n is a natural number and if A is a G-module such that $H^i(H, A) = 0$ for $i = 1, 2, \cdots, n-1$, then the sequence

$$0 \to H^n(G/H, A^H) \xrightarrow{\lambda} H^n(G, A) \xrightarrow{\rho} H^n(H, A)^G$$
$$\xrightarrow{\tau} H^{n+1}(G/H, A^H) \xrightarrow{\lambda} H^{n+1}(G, A)$$

is exact, where M^G with a G-module M denotes the submodule of M consisting of all G-invariant elements of M and where λ, ρ and τ denote lift, restriction and transgression maps respectively. Dually we have the theorem of fundamental exact sequences in homology, which in case of a finite group G may be formulated in terms of negative dimensional cohomology groups as follows: If $n \geqq 2$ and if $H^{-i}(H, A) = 0$ for $i = 2, 3, \cdots, n-1$, then we have an exact sequence

$$0 \leftarrow H^{-n}(G/H, A_H) \leftarrow H^{-n}(G, A) \leftarrow H^{-n}(H, A)_G$$
$$\leftarrow H^{-(n+1)}(G/H, A_H) \leftarrow H^{-(n+1)}(G, A)$$

Received September 22, 1956.