FINITE DIMENSIONALITY OF CERTAIN TRANSFORMATION GROUPS

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1. Introduction

The main result to be proved in this paper is as follows:

THEOREM A. *If G is a locally compact effective transformation group of a manifold M, then G is finite-dimensional.*

By a manifold M is meant a separable, metric, connected, locally euclidean space. For the proof of Theorem A it will be sufficient to consider the case where G is compact. This is because an infinite-dimensional locally compact group contains an infinite-dimensional compact subgroup [4, 5].

This theorem gives no information on whether or not G must be a Lie group. Information on this latter question depends on an analysis of the case where G is zero-dimensional.

The proof of Theorem A will be based on Theorem B which is known for compact Lie groups (see [7] and for extensions [6] and [8]).

THEOREM B. *Let G be a compact connected group which acts on an n-dimensional manifold M and let F be the set of points of M left fixed by every element of G. If* $\dim F \geqq n - 1$, *then* $F = M$.

2. Reduction of Theorem A to Theorem B

It has been shown in the introduction that for the proof of Theorem A, the group G may be assumed compact. It is therefore now assumed that G is a compact effective transformation group of a manifold M.

Let G^* be the identity component of G, and let k be the highest dimension of any orbit of G^*. The set of points of M which lie on k-dimensional orbits is an open set and a component of this open set will be denoted by Y. The group G^* acts as a transformation group of Y,

$$G^*(Y) = Y,$$

although it may, conceivably, not be effective. Let H be the subgroup of G^* which leaves every point of Y fixed. Then H is an invariant subgroup of G^*, and G^*/H acts in a natural way on Y and the action is effective. It is known [5, p. 243] that G^*/H is finite-dimensional.

Assuming that Theorem B is true, it follows that H^* leaves all of M fixed. But G was taken effective so H^* can contain only the identity, and H is zero-dimensional. Then

$$\dim G = \dim G^* = \dim G^*/H,$$

and G is finite-dimensional.

This completes the proof that Theorem B implies Theorem A.

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