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QUASIMODES AND RESONANCES: SHARP LOWER BOUNDS

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1. Introduction. The purpose of this paper is to obtain sharp lower bounds of the number of resonances (scattering poles) close to the real axis. We consider a situation where one can construct real *quasimodes*, that is, a sequence of approximate real "resonances" and corresponding approximate solutions supported in a fixed compact set. Our main result states, loosely speaking, that quasimodes are perturbed resonances near the real axis and that the number of resonances close to the real axis is at least equal to that of the quasimodes, counting multiplicities.

Quasimode constructions with polynomially small errors have been known for a long time in various situations (see, e.g., [C], [R], [L], [P1], and [CP]; see also [P2] for a construction with an exponentially small error). An open problem, however, for problems in unbounded domains, was whether the mere fact that one can construct quasimodes implies existence of resonances close to them. In particular, it was not known whether an elliptic periodic trapped ray in obstacle scattering generated a sequence of resonances converging to the real axis, although a construction of quasimodes in this case was available. The first result in this direction appeared in [StV2, Lemma 1] in the study of resonances caused by Rayleigh surface waves in linear elasticity. It was shown, for general compactly supported perturbations in odd dimensional spaces, that existence of real quasimodes with polynomially small error implied existence of resonances converging to the real axis at the same rate. An important role in the proof of that lemma was played by an a priori exponential estimate on the cutoff resolvent, established by Zworski (see the remarks before Lemma 1 in Section 3). The method in [StV2], however, was not sensitive enough to obtain information on the density of those resonances; it could only prove their existence. An asymptotic formula for the Rayleigh resonances for the specific problem studied in [StV1] and [StV2] for convex obstacle was obtained in [SjV].

A major step ahead was made by Tang and Zworski [TZ], who considered any space dimension and nonnecessarily compactly supported perturbations. They observed that one can localize not only near the real axis as done in [StV2, Lemma 1], but also near a quasimode to obtain that if the quasimode is large enough, then there is always a resonance close to it. This confirmed the expectation that quasimodes are perturbed resonances. The results in [TZ] also imply lower bounds on the number of resonances near the real axis. For any known construction, we get at least linear

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