Vol. 98, No. 3 DUKE MATHEMATICAL JOURNAL © 1999

SEPARATION PROPERTIES OF THETA FUNCTIONS

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0. Introduction. Let X be a nonsingular, connected, projective curve defined over an algebraically closed field k. Let U^s denote the set of isomorphism classes of stable vector bundles on X with given degree d and rank r. In the 1960s, C. S. Seshadri and D. Mumford ([15], [18], and [19]) supplied U^s with a natural structure of quasi-projective variety, together with a natural compactification U, by adding semistable vector bundles at the boundary. The method used in the construction of such a structure was Mumford's then recently developed geometric invariant theory (GIT); see [8]. Roughly, the method consists of producing a variety R and an action of a reductive group G on X, linearized at some ample invertible sheaf L on R, such that $U = R/G$ set-theoretically. Then the GIT tells us how to supply R/G with a natural scheme structure obtained from the G-invariant sections of tensor powers of L.

Until recently, Seshadri and Mumford's construction was the only purely algebraic construction available. In 1993, Faltings [7] showed how to construct U^s and its compactification U, avoiding the GIT. His method, described also in [21], consisted of considering the so-called theta functions on R, that are naturally defined provided R admits a family with the so-called local universal property. (We observe that the theta functions considered in this paper are just those associated with vector bundles on X, as is clear from our definition in Section 2. Beauville [3, Section 2] has a more encompassing definition of theta functions than ours.) The theta functions are in fact G-invariant sections of tensor powers of a certain G-linear invertible sheaf L'_θ on R. Roughly speaking, using his first main lemma [21, Lemma 3.1], Faltings showed that there are enough theta functions to produce a G-invariant morphism $\theta : R \longrightarrow \mathbf{P}^N$. By semistable reduction, the image

$$U_\theta := \theta(R) \subseteq \mathbf{P}^N$$

is a closed subvariety. Since θ is G-invariant, then θ factors (set-theoretically) through a map $\pi : U = R/G \longrightarrow U_\theta$. Then, using his second main lemma [21, Lemma 4.2], Faltings showed that there is a bijection between the normalization of U_θ and U, through which U acquires a natural structure of projective variety.

There are interesting consequences of Faltings's work besides the construction

Received 22 October 1997. Revision received 19 June 1998.
1991 *Mathematics Subject Classification.* Primary 14H60.
Author's research supported by a Massachusetts Institute of Technology Japan Program Starr fellowship, Programa de Apoio a Núcleos de Excelência do Ministério do Ciência e Tecnologia, and Conselho Nacional de Desenvolvimento Científico e Tecnológico Processo 300004/95-8 (NV).