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CENTRAL VALUES OF HECKE L-FUNCTIONS OF CM NUMBER FIELDS

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0. Introduction. It is well known that the zeta function of CM (complex multiplication) abelian varieties can be given in terms of L-functions of associated Hecke characters. In this paper, we prove a formula expressing the central special value of the L-function of certain Hecke characters in terms of theta functions. The formula easily implies that the central value is nonnegative and yields a criterion for its positivity. Combining this criterion with the work of Arthaud and Rubin, we show that certain CM elliptic curves have Mordell-Weil rank zero over their field of definition.

Let F be a totally real number field of degree t and let μ be a quadratic Hecke character of F of conductor $\mathfrak{f}$ such that $(2\mathcal{O}_F, \mathfrak{f}) = 1$. Given a CM extension E of F, we consider the twist $\chi = \chi_{\mathrm{can}}\tilde{\mu}$ of μ by a "canonical" Hecke character χ_{can} of E (see Section 2), where $\tilde{\mu} = \mu \circ N_{E/F}$, as well as its odd powers χ^{2k+1}, $k \in \mathbb{Z}_{\geq 0}$. Consider the following condition:

(∗) All units of E are real and every prime of F dividing $2\mathfrak{f}$ is split in E/F.

Our main result is the following.

THEOREM 0.1 (Sketch of Theorem 2.5). *Assume that F has ideal class number* 1 *and* $(-1)^{kt}\mu_\infty(-1) = 1$, *where* $\mu_\infty : (F \otimes \mathbb{R})^* \longrightarrow \mathbb{C}^*$ *is the infinite part of* μ. *Then there is an explicit theta function* $\theta_{\mu,k}$ *over* F, *depending only on* μ *and* k, *such that for every CM quadratic extension* E *of* F *satisfying the condition* (∗), *the central L-value*

$$L\left(k+1, \chi^{2k+1}\right) = \kappa \left| \sum_{C \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\mathfrak{A})}{\chi^{2k+1}(\bar{\mathfrak{A}})} \right|^2 . \tag{0.1}$$

Here, κ *is an explicit positive number,* $\mathfrak{A} \in C^{-1}$ *is a primitive ideal relatively prime to* $2\mathfrak{f}$, *and* $\theta_{\mu,k}(\mathfrak{A})$ *is essentially the value of a theta function* $\theta_{\mu,k}$ *at a CM point in* E *associated to* $\mathfrak{A}^2$.

We emphasize that $\theta_{\mu,k}$ is *independent* of the CM field E, which is one of the

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