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CANONICAL PERIODS AND CONGRUENCE FORMULAE

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0. Introduction. The purpose of this article is to show how congruences between the Fourier coefficients of Hecke eigenforms give rise to corresponding congruences between the algebraic parts of the critical values of the associated L-functions. This study was initiated by Mazur in his fundamental work on the Eisenstein ideal (see [Maz1] and [Maz2]), where it was made clear that congruences for analytic L-values are closely related to the integral structure of certain Hecke rings and cohomology groups. The results of [Maz2] also showed that congruences are useful in the study of the nonvanishing of L-functions. This idea was then further developed by Stevens [Ste1] and Rubin and Wiles [RW]. The work of Rubin and Wiles, in particular, used congruences to study the behavior of elliptic curves in towers of cyclotomic fields. A key ingredient there was a theorem of Washington, which states, roughly, that almost all L-values in certain families are nonzero modulo p.

This theme was recently taken up again in the work of Ono and Skinner [OS1], [OS2], James [Jam], and Kohnen [Koh]. While the earlier history was primarily concerned with cyclotomic twists, the current emphasis is on families of twists by quadratic characters. Here one wants quantitative estimates for the number of quadratic twists of a given modular form which have nonvanishing L-function at $s = 1$. We continue this trend in the present work by using our general results to obtain a strong nonvanishing theorem for the quadratic twists of modular elliptic curves with rational points of order 3. This generalizes a beautiful example due to James [Jam] and provides new evidence for a conjecture of Goldfeld [Gol]. It should, however, be pointed out that even the study of quadratic twists may be traced back to Mazur: the reader is urged to look at pages 212–213 of [Maz2] and especially at the footnote at the bottom of page 213. The theorems of Davenport and Heilbronn [DH] and Washington [Was], which are crucial in this paper, are both mentioned in Mazur's article.

We begin by discussing the congruences that lie at the heart of this article. Thus, let $f = \sum a_n q^n$ be an elliptic modular cuspform of level M and weight $k \geq 2$. Assume that f is a simultaneous eigenform for all the Hecke operators and that $a_1(f) = 1$. The L-function associated to f is defined by the Dirichlet series $L(s, f) = \sum a_n n^{-s}$, which converges for the real part of s sufficiently large and has analytic continuation to $s \in \mathbf{C}$. A fundamental theorem of Shimura [Shi] states that $L(s, f)$ enjoys the following algebraicity property.

THEOREM 0.1 (Shimura). *There exist complex periods $\Omega_f^{\pm}$ such that for each in-*

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