Vol. 94, No. 3 DUKE MATHEMATICAL JOURNAL

DIFFERENTIAL INVARIANTS OF CLASSICAL GROUPS

XIAOPING XU

1. Introduction. Symmetry is an important feature of nature. Many physical phenomena have been described by differential equations, which are invariant under the action of certain groups. The symmetry of a differential equation often leads one to find certain nice exact solutions (e.g., cf. [FSS], [O1], [O2]). It has also been a major method in the study of complete solvability.

There have been many examples of finding the symmetry group of a given differential equation and then using it to find exact solutions (e.g., cf. [AKO], [FSS], [O1], [O2]). Special types of differential equations with a prescribed symmetry group have been studied up to a certain degree (e.g., cf. [FSS]). Second-order differential invariants of the rotation group and its extensions over the fundamental representation were found by Fushchich and Yegorchenko (cf. [FY]). Mathematically, it is desirable to find all the partial differential equations with a prescribed symmetry group, such as the Galilei group and the Poincaré group. It seems to us that there has not been a systematic study in this direction. (The problem was also mentioned by Olver in [O2].) In fact, this is, in general, a very difficult problem, because the "jet spaces" are infinite-dimensional and the invariants on them are not easy to study. The main purpose of this paper is to solve this problem over the fundamental representations of classical groups $SL(n)$, $SO(m,n)$, $SP(2n)$, $U(n)$ and their semiproducts with the translations. As special cases, we have found all the differential invariants of the Galilei group and the Poincaré group. Let us give a more detailed description.

Throughout this paper, we denote by $\mathbf{R}$ the field of real numbers and by $\mathbf{N}$ the set of natural numbers $\{0,1,2,\ldots\}$. Moreover, we denote $\mathbf{N}^+ = \mathbf{N}\backslash\{0\}$. Let $m, n \in \mathbf{N}^+$, and suppose that

$$X = \mathbf{R}^m = \{(a_1,\ldots,a_m) \mid a_i \in \mathbf{R}\} \tag{1.1}$$

is the configuration space of some physical entity and $U = \mathbf{R}^n$ is the target space. We denote by $(x_1,\ldots,x_m)$ and $(u_1,\ldots,u_n)$ the coordinate functions of X and U, respectively. Furthermore, we assume that $u_1,\ldots,u_n$ are C^∞-differentiable functions in $\{x_1,x_2,\ldots,x_m\}$. Let

$$\Gamma = \mathbf{N}^m = \{\alpha = (\alpha_1,\ldots,\alpha_m) \mid \alpha_i \in \mathbf{N}\}, \quad \varepsilon_i = (0,\ldots,0,\overset{i}{1},0,\ldots,0). \tag{1.2}$$

Received 2 April 1997. Revision received 20 May 1997.
1991 *Mathematics Subject Classification.* Primary 53A55; Secondary 35A30, 15A72.
Research supported by Hong Kong Research Grant Council Competitive Earmarked Research grant HKUST709/96P.