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COMMUTATOR COVERINGS OF SIEGEL THREEFOLDS

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0. Introduction. In this paper we study cusp forms of small weight with respect to the paramodular group Γ_t and Siegel modular threefolds. Since cusp forms of weight 3 define canonical differential forms on the moduli space $\mathscr{A}_t$ of $(1,t)$-polarized abelian surfaces, the existence or nonexistence of such forms has important geometric consequences.

For $t \geqslant 1$ the paramodular group Γ_t is not a maximal discrete group. One can add to it a number of exterior involutions to obtain a maximal normal extension Γ_t^* such that Γ_t^*/Γ_t is a product of $\mathbb{Z}_2$-components. We then have the tower of Siegel threefolds

$$\begin{array}{ccccccccc} \mathscr{A}_t^{com} & \longrightarrow & \mathscr{A}^{(l)} & \longrightarrow & \mathscr{A}_t & \longrightarrow & \mathscr{A}^{(r)} & \longrightarrow & \mathscr{A}_t^* \\ \| & & \| & & \| & & \| & & \| \\ \Gamma_t'\backslash\mathbb{H}_2 & \longrightarrow & \Gamma^{(l)}\backslash\mathbb{H}_2 & \longrightarrow & \Gamma_t\backslash\mathbb{H}_2 & \longrightarrow & \Gamma^{(r)}\backslash\mathbb{H}_2 & \longrightarrow & \Gamma_t^*\backslash\mathbb{H}_2, \end{array} \tag{0.1}$$

where $\Gamma_t' \subset \Gamma^{(l)} \subset \Gamma_t \subset \Gamma^{(r)} \subset \Gamma_t^*$ and Γ_t' is the commutator subgroup of Γ_t. We call these varieties *commutative neighbours* of $\mathscr{A}_t$; they have very interesting properties. Neighbours $\mathscr{A}^{(r)}$ of spaces to the right of $\mathscr{A}_t$ (i.e., finite quotients) were studied in [GH2]. The coverings in (0.1) to the left of $\mathscr{A}_t$ are also galois with a finite abelian Galois group. This paper is devoted to neighbours $\mathscr{A}^{(l)}$ to the left of $\mathscr{A}_t$.

Recall that the geometric genus of $\mathscr{A}_t$ can be zero only for twenty exceptional polarizations $t = 1, \dots, 12$, 14, 15, 16, 18, 20, 24, 30, 36 (see [G1]). Hypothetically, for all of them $\mathscr{A}_t$ is rational or unirational. (See [GP] for an announcement of some results in this direction.) Hence no weight-3 cusp forms should exist for these values of t. (For an easy proof of this fact for $t \leqslant 8$, see Corollary 3.3.) On the other hand, we construct many examples of cusp forms of small weight ($k \leqslant 3$) with respect to Γ_t with a character. It follows that $\mathscr{A}_t$ usually has a double modular covering with positive geometric genus. One of the main results of this paper is the following theorem.

THEOREM 0.1. *Let $\tilde{\mathscr{A}}_t^{com}$ be a smooth projective model of the maximal abelian covering $\mathscr{A}_t^{com} = \Gamma_t'\backslash\mathbb{H}_2$ of $\mathscr{A}_t$. Then*

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