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THE BINOMIAL FORMULA FOR NONSYMMETRIC MACDONALD POLYNOMIALS

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1. Introduction. The q-binomial theorem [GR] is essentially the expansion of $(x-1)(x-q)\cdots(x-q^{k-1})$ in terms of the monomials x^d. In a recent paper [Ok], Okounkov has proved a beautiful multivariate generalization of this in the context of symmetric Macdonald polynomials [M1]. These polynomials have nonsymmetric counterparts [M2] that are of substantial interest; in this paper, we establish nonsymmetric analogues of Okounkov's results.

An integral vector $v \in \mathbb{Z}^n$ is called "dominant" if $v_1 \geqslant \cdots \geqslant v_n$; it is called a "composition" if $v_i \geqslant 0$ for all i. To avoid ambiguity, we reserve the letters u, v for integral vectors, α, β, γ for compositions, and λ, μ for "partitions" (dominant compositions).

We write $|v|$ for $v_1 + \cdots + v_n$, and denote by w_v the (unique) shortest permutation in the symmetric group S_n such that $v^+ = w_v^{-1}(v)$ is dominant. Let $\mathbb{F}$ be the field $\mathbb{Q}(q,t)$ where q,t are indeterminates. We write $\tau = (1, t^{-1}, \ldots, t^{-n+1})$ and define $\bar{v} = \bar{v}(q,t)$ in $\mathbb{F}^n$ by

$$\bar{v}_i = q^{v_i}(w_v\tau)_i.$$

Inhomogeneous analogues of nonsymmetric Macdonald polynomials were introduced in [Kn] and [S3]. They form an $\mathbb{F}$-basis for $\mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]$, and are defined as follows.

Definition. $G_\alpha \equiv G_\alpha(x; q, t)$ is the unique polynomial of degree $\leqslant |\alpha|$ in $\mathbb{F}[x]$ such that

(1) the coefficient of $x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in G_α is 1;

(2) G_α vanishes at $x = \bar{\beta}$, for all compositions $\beta \neq \alpha$ such that $|\beta| \leqslant |\alpha|$.

As shown in Theorem 3.9 of [Kn], the top homogeneous part of G_α is the nonsymmetric Macdonald polynomial E_α for the root system A_{n-1} (see [M2] and [C]). Moreover, by Theorem 4.5 of [Kn], we have $G_\alpha(\bar{\beta}) = 0$ unless "$\alpha \subseteq \beta$." Here $\alpha \subseteq \beta$ means that if we write $w = w_\beta w_\alpha^{-1}$, then $\alpha_i < \beta_{w(i)}$ if $i < w(i)$ and $\alpha_i \leqslant \beta_{w(i)}$ if $i \geqslant w(i)$.

In this paper, we obtain several new results about the polynomials G_α. Our first result is a formula for the special value $G_\alpha(a\bar{0}) = G_\alpha(a\tau) \in \mathbb{F}[a]$, where a is an indeterminate. This can be described in the following manner. We identify α

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