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ACTION-MINIMIZING MEASURES AND THE GEOMETRY OF THE HAMILTONIAN DIFFEOMORPHISM GROUP

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1. Introduction. Hamiltonian systems have always been a main subject of interest in the field of dynamical systems. Since the development of symplectic topology in the 1980s, they are being studied also from a more geometric viewpoint. Namely, due to Hofer, the group of Hamiltonian diffeomorphisms can be equipped with an intrinsic bi-invariant metric. Although its geometry is the subject of intensive research (see, e.g., [BP2], [Ho], [HZ], [LM], and [Si2]), it is only partially understood, and many fundamental questions are still unanswered. The motivation for this work came from the wish to find connections between those two branches, the classical dynamical and the modern geometric.

A distinguished class of Hamiltonian systems is given by mechanical systems with the phase space being the cotangent bundle of some closed manifold (often the n-dimensional torus $\mathbb{T}^n$) and the Hamiltonian being the sum of kinetic and potential energy. An essential feature of such systems is the convexity of the Hamiltonian with respect to the fibre variables, which yields a correspondence between Hamiltonian and Lagrangian systems via the Legendre transform.

In the case of one degree of freedom, where the phase space is the 2-dimensional cylinder, Moser [Mo] showed that the class of diffeomorphisms which can be generated by a fibrewise convex Hamiltonian comprises, in particular, the class of monotone twist mappings. The latter has been thoroughly studied. For small perturbations of integrable maps, for instance, KAM-theory guarantees the existence of invariant circles; according to Aubry-Mather theory, these invariant circles do not disappear completely when the perturbation becomes large, but disintegrate into invariant Cantor-type sets.

For higher dimensions, Mather [Ma2] developed a generalization of Aubry-Mather theory for time-1 maps of convex Lagrangian functions, based on so-called action-minimizing measures. Its framework is as follows. Consider fibrewise convex Lagrangians L on $\mathbb{S}^1 \times T\mathbb{T}^n$ that are 1-periodic in time and probability measures invariant under the flow associated to L. Each such measure μ possesses a rotation vector $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R})$. An action-minimizing measure is one that minimizes the action

$$\int L\, d\mu$$

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