EXPLICIT ELLIPTIC UNITS, I

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Elliptic units are units in abelian extensions of an imaginary quadratic field $K \subset \mathbb{C}$ that are obtained as values of certain modular functions at points in $K \cap \mathscr{H}$ (with $\mathscr{H}$ denoting the upper half-plane). In this series of papers, we are concerned with those elliptic units that are expressible as ratios of values of the Dedekind eta function η. In particular, we will describe explicitly how the absolute Galois group of K acts on these units. This action is completely characterized by the Shimura reciprocity law [Sh], but the calculations are greatly complicated by the presence of 24th roots of unity in the transformation formulas for η.

The subject of values of particular modular functions at points in $K \cap \mathscr{H}$—the description of their algebraic and Galois properties, the explicit determination of their minimal polynomials, and so on—is very old. We have not attempted the monumental task of sorting out the history of these *singular moduli*, as they were classically known, or that of providing a full bibliography. Instead, we will just mention the following.

In his 1828 paper on elliptic functions [Ab, *Oeuvres*, pp. 380–382], Abel proves, among many other things, the following identity, here written in modern notation:

$$\frac{\eta^2(\sqrt{-5}/2)}{2\eta^2(2\sqrt{-5})} = \frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}.$$

(The left-hand side is an example of an elliptic unit; its minimal polynomial is $x^4 - 2x^3 - 2x^2 - 2x + 1$.)

In the 1930s, G. N. Watson computed the minimal polynomials of a number of ratios of values of η by an ad hoc, trial and error process for the selection of the appropriate 24th roots of unity. He managed to calculate in this way, working by hand with expansions of up to 10 decimal places, many examples of degrees less than 20. In the third paper of his *Singular Moduli* series, he remarks [Wa, p. 89 footnote], "I have dealt successfully with $n = 479$ and $n = 599$; for each of these values of n, $h = 25$ [the degree of the equation] and the selection has to be made from $3^{12} = 531,441$ values. Each of the corresponding computations required twelve hours' work." Weber's book [We] contains many of these polynomials for degrees less than 10.

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