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LIMIT DISTRIBUTION OF SMALL POINTS ON ALGEBRAIC TORI

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1. Introduction. We denote by $\mathbf{C}^*$ the multiplicative group of complex numbers, by $\bar{\mathbf{Q}}$ the field of all algebraic numbers, and by $\bar{\mathbf{Q}}^*$ its multiplicative group. Let δ_α be the Dirac measure at $\alpha \in (\mathbf{C}^*)^N$, that is, the probability measure on $(\mathbf{C}^*)^N$, supported[1] at $\{\alpha\}$. Also, let ν be the probability measure on $(\mathbf{C}^*)^N$, supported at the unit polycircle $|z_1| = \cdots = |z_N| = 1$, where it coincides with the normalized Haar measure.

Recall that a sequence $\{\mu_k\}$ of probability measures on a metric space S *weakly converges to* μ (notation: $\mu_k \xrightarrow{w} \mu$) if for any bounded continuous function $f: S \to \mathbf{R}$ we have $(f, \mu_k) \to (f, \mu)$ as $k \to \infty$.

A sequence $\{\alpha_k\}$ of points in $(\bar{\mathbf{Q}}^*)^N$ is *strict* if any proper algebraic subgroup of $(\bar{\mathbf{Q}}^*)^N$ contains α_k for only finitely many values of k.

Given $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(N)}) \in (\bar{\mathbf{Q}}^*)^N$, we denote by $h(\alpha)$ its *absolute logarithmic height*:

$$h(\alpha) = [\mathbf{K} : \mathbf{Q}]^{-1} \sum_v [\mathbf{K}_v : \mathbf{Q}_v] \max(0, \log|\alpha^{(1)}|_v, \ldots, \log|\alpha^{(N)}|_v), \tag{1}$$

where $\mathbf{K}$ is a number field containing $\alpha^{(1)}, \ldots, \alpha^{(N)}$, and the summation is extended to all valuations of $\mathbf{K}$, normalized so that their restrictions to $\mathbf{Q}$ define usual infinite or p-adic valuations. It is well known [6, Section 3.1] that the sum in (1) does not depend on the particular choice of $\mathbf{K}$.

Similarly, given $\alpha \in (\bar{\mathbf{Q}}^*)^N$, we define a probability measure $\bar{\delta}_\alpha$ on $(\mathbf{C}^*)^N$ by

$$\bar{\delta}_\alpha = [\mathbf{K} : \mathbf{Q}]^{-1} \sum_{\sigma: \mathbf{K} \hookrightarrow \mathbf{C}} \delta_{\sigma(\alpha)},$$

where $\mathbf{K}$ is as in the previous paragraph, and the summation is extended to all distinct complex embeddings of $\mathbf{K}$. Again, the sum is independent upon the particular choice of $\mathbf{K}$.

THEOREM 1.1. *Let $\{\alpha_k\}$ be a strict sequence of points in $(\bar{\mathbf{Q}}^*)^N$ with $h(\alpha_k) \to 0$. Then $\bar{\delta}_{\alpha_k} \xrightarrow{w} \nu$.*

This theorem was inspired by recent works of Szpiro, Ullmo, and Zhang (see [14] and [18]), who obtained a similar result for small points on abelian varie-

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[1] We say that a probability measure μ is supported at U if $\mu(U) = 1$.