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ROOT NUMBERS AND ALGEBRAIC POINTS ON ELLIPTIC SURFACES WITH BASE $\mathbb{P}^1$

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Let C be a nonsingular projective curve defined over a number field K. By *elliptic fibration* defined over K, with base C, we will mean a projective variety $\mathscr{E}$ defined over K with a K-morphism π to C such that all but finitely many fibers are nonsingular curves of genus 1. If π has a section (defined over K), we will call $\mathscr{E}$ an *elliptic surface* (defined over K). By [11, Chapter III, Proposition 3.8], we can think of an elliptic surface defined over K and with base C as an elliptic curve defined over $K(C)$, where $K(C)$ denotes the function field of C/K.

Let $\mathscr{E}$ be an elliptic surface defined over a number field K with base curve C of genus $\leqslant 1$. It is unknown whether there exists, in general, a finite extension L of K such that $\mathscr{E}(L)$ is Zariski dense in $\mathscr{E}$. It had been conjectured, for the more general case of elliptic fibrations with base curve of genus $\leqslant 1$, that the answer is positive (for discussion of this and related problems, the reader is referred to Abramovich [1] and Silverman [10, p. 225]); but recently Colliot-Thélène, Skorobogatov, and Swinnerton-Dyer [2] have given a counterexample. Their example is, in our terminology, an elliptic fibration, but not an elliptic surface, since it has no section. In this paper we will provide strong evidence that the answer is positive in the case of elliptic surfaces with non-constant j-invariant and base the projective line. We will prove several results about root numbers of fibers of such elliptic surfaces. From them it will follow that the conjectures of Birch-Swinnerton-Dyer (BSD) and Deligne-Gross (DG), taken together, imply that it will always be the case, for elliptic surfaces with base the projective line and nonconstant j-invariant, that a finite extension L of K exists for which $\mathscr{E}(L)$ is Zariski dense in $\mathscr{E}$.

One way to show that such an L exists is to find, for each $\mathscr{E}$, an extension L/K, such that $E_P(L)$ has nonzero rank for infinitely many fibers E_P, $P \in C(L)$. For any elliptic curve E over a number field L, we have the well-known implication of BSD,

$$W(E/L) = (-1)^{\operatorname{rank} E(L)}, \tag{1}$$

where $W(E/L)$ is the root number associated to E over L; this root number is a sign occurring in an appropriate normalization of the conjectural functional equation of the L-function of E over L, but it also has an intrinsic definition, independent of conjectures, as a product of local root numbers (for general background on

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