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GEODESIC INTERSECTIONS IN ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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Section 0. Introduction. By a *hyperbolic n-manifold*, we shall mean a complete, orientable, n-dimensional Riemannian manifold, all of whose sectional curvatures are -1. If M is a hyperbolic 3-manifold, the universal cover of M can be identified with $\mathbb{H}^3$, the upper half-space model of hyperbolic 3-space, and M is realized as $\mathbb{H}^3/\Gamma$ for some Γ a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Now $\mathrm{Isom}^+(\mathbb{H}^3)$ can be identified with $\mathrm{PSL}(2, \mathbb{C})$ (which, in turn, is isomorphic to $\mathrm{PGL}(2, \mathbb{C})$), and Γ is called a *Kleinian group*. In the sequel we will only be interested in the case when M is closed, in which case Γ is referred to as *cocompact*.

Of interest to us is the structure of the set of closed geodesics in closed hyperbolic 3-manifolds. The motivation comes from the following. A closed geodesic in a (closed) hyperbolic n-manifold is simple if it has no self-intersections and is nonsimple otherwise. In dimension 2, every closed hyperbolic manifold has a nonsimple closed geodesic. However, in dimension 3, the situation is much more complex. Many closed hyperbolic 3-manifolds contain immersions of totally geodesic surfaces, and so there are nonsimple closed geodesics. In [JR], examples were given of closed hyperbolic 3-manifolds containing a nonsimple closed geodesic but having no immersed totally geodesic surface. However, it was shown in [CR] that there exist closed hyperbolic 3-manifolds in which all closed geodesics are simple. Subsequently, Basmajian and Wolpert [BW] showed that almost all 3-manifolds arising as the quotient of $\mathbb{H}^3$ by a quasi-Fuchsian subgroup of $\mathrm{PSL}(2, \mathbb{C})$ have all closed geodesics simple and disjoint. It was shown in [CR] that closed geodesics of the same complex length (see Lemma 2.2 for the definition) were disjoint. The natural conjecture motivated by this was that the examples constructed in [CR] had all closed geodesics disjoint.

The main results here show that this conjecture is both almost true (they have no geodesics that intersect, except at right angles) and spectacularly false (any pair of closed geodesics admits infinitely many closed geodesics that intersect both geodesics of the pair perpendicularly). The latter statement is shown to be true for all closed arithmetic hyperbolic 3-manifolds. Like the methods of [CR] and [JR], the methods here rely heavily on arithmetic techniques.

In the final section of the paper, we apply this same technology to arrive at a partial answer to a question posed by Weeks: Does every finite-volume hyper-

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