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ON THE VOLUMES OF COMPLEX HYPERBOLIC MANIFOLDS

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1. Introduction. Let M be a complete locally symmetric Riemannian manifold of negative curvature. The main goal of this paper is to give estimates on the smallest volumes of such M's. We will do that in the complex case, but quaternionic or octonion analogues of Theorem 5.1 are true.

Fix $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{Ca}\}$ the real, complex, or quaternionic field, or the Cayley octonion algebra, and $n \geqslant 2$ an integer with $n = 2$ if $\mathbf{K} = \mathbf{Ca}$. By a theorem of E. Cartan, the universal cover of M is isometric to $\mathbf{H}_{\mathbf{K}}^n$, the hyperbolic space over $\mathbf{K}$ of dimension n. By a theorem of H. C. Wang (see [Wan, Theorem 8.1]), if $(\mathbf{K}, n) \neq (\mathbf{R}, 2), (\mathbf{R}, 3)$, and of Jorgensen-Thurston (see [Gro2]) if $(\mathbf{K}, n) = (\mathbf{R}, 3)$, there does exist a manifold covered by $\mathbf{H}_{\mathbf{K}}^n$ of smallest volume. Moreover, the minimum is obtained by only finitely many manifolds (up to isometry).

The *closed* real hyperbolic 3-manifold of smallest known volume is the J. Weeks and S. Matveev-A. Fomenko manifold, having volume ≈ 0.94272. The best-known lower bound is ≈ 0.00115, due to F. Gehring-G. Martin [GM]. Of related interest is the work of M. Culler-P. Shalen (with P. Wagreich, J. Anderson-R. Canary, S. Hersonsky) proving, for example, that every real hyperbolic 3-manifold of smallest volume has first Betti number less than or equal to 2 [CHS]. Note that the smallest volume of a noncompact real hyperbolic 3-manifold and 3-orbifold are, respectively, σ (C. Adams [Ada]) and $\sigma/24$ (R. Meyerhoff [Mey1]), where σ is the volume of the regular ideal real hyperbolic tetrahedra, $\sigma \approx 1.0149414$.

If the real dimension of M is even and if M has finite volume, since $\mathbf{H}_{\mathbf{K}}^n$ is homogeneous, the Gauss-Bonnet formula (see [Spi, vol. 4, page 443]; as extended by Harder-Gromov [Gro3, page 84] in the noncompact case; see also [Mum1]; or Hirzebruch proportionality theorem [Hir3, Theorem 22.2.1]) tells us that there is a constant $\kappa_{\mathbf{K},n}$ such that

$$\operatorname{vol}(M) = \kappa_{\mathbf{K},n}\, \chi_{\mathrm{top}}(M)$$

with $\chi_{\mathrm{top}}(M)$ the Euler characteristic of M. The exact value of the constant has been explicitly computed, for instance in [Hir1], giving in the complex case, when the holomorphic sectional curvature is normalized to be -1 (hence the sectional

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