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THE LAPLACIAN ON RAPIDLY BRANCHING TREES

KOJI FUJIWARA

1. Introduction and the statement of main results. Let M be a complete Riemannian manifold and Δ the Laplacian on M. Fix a point $p \in M$ and write $K(r) = \sup\{K(x,\pi) | d(p,x) \geqslant r\}$ where π is a two plane in $T_x M$ and K denotes the sectional curvature. H. Donnelly and P. Li showed the following [DL].

THEOREM (Donnelly, Li). *Let M be a complete simply connected negatively curved Riemannian manifold. If $\lim_{r\to\infty} K(r) = -\infty$, then Δ has no essential spectrum.*

In this article we show an analogous theorem for graphs.

Let $G = G(V,E)$ be a locally finite, infinite graph with the set of vertices V and the set of directed edges E. The Laplacian $\Delta = \Delta_G$ is given by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{x\sim y} (f(x) - f(y)),$$

where $x \sim y$ means x and y are joined by an edge and $m(x)$ is the *multiplicity* at x defined by $\sharp\{y | x \sim y\}$. The domain of Δ, $D(\Delta)$, is $L^2(V)$ with its natural L^2-structure: $L^2(V) = \{f: V \to \mathbb{R} \mid (f,f) = \sum_{x\in V} m(x) f^2(x) < \infty\}$, $(f,g) = \sum_{x\in V} m(x) f(x) g(x)$. $L^2(V) = \overline{C_0(G)}$, where $C_0(G)$ is the set of functions on V with a finite support. We sometimes write $L^2(G)$ instead of $L^2(V)$. Put $L^2(E) = \{\phi: E \to \mathbb{R} \mid \phi([x,y]) = -\phi([y,x]), (\phi,\phi) = 1/2 \sum_{e\in E} \phi^2(e) < \infty\}$ and $(\phi,\psi) = 1/2 \sum_{e\in E} \phi(e)\psi(e)$, where $[x,y]$ is an edge from x to y. The coboundary operator $L^2(V) \to L^2(E)$ is $df([x,y]) = f(x) - f(y)$ and we have $\Delta f = \delta df$, $(\Delta f, g) = (df, dg)$ for $f, g \in L^2(V)$, where δ is the adjoint operator of d. Δ is a selfadjoint operator with $0 \leqslant \Delta \leqslant 2$.

Let S be a finite set of vertices. Put $\partial S = \{(x,y) | x \notin S, y \in S, x \sim y\}$, $L(\partial S) = \sharp \partial S$, and $A(S) = \sum_{x\in S} m(x)$. The *isoperimetric constant* is given by $\alpha(G) = \inf_S \{L(\partial S)/A(S)\}$. The *isoperimetric constant at infinity*, α_∞, is defined by $\alpha_\infty = \lim_K \alpha(G - K)$, where K runs over all finite subsets. We have $0 \leqslant \alpha \leqslant \alpha_\infty \leqslant 1$. We denote the spectrum of Δ by $\mathrm{Spec}(\Delta)$ and the essential spectrum by $\mathrm{Ess}(\Delta)$.

THEOREM 1. $\alpha_\infty = 1$ *if and only if* $\mathrm{Ess}(\Delta) = \{1\}$.

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