SMOOTHING ESTIMATES FOR NULL FORMS AND APPLICATIONS

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1. Introduction. This is a follow-up on our work [KlMa] concerning space-time estimates for null forms and their relevance to the problem of optimal regularity of initial conditions for which the corresponding initial value problem (I.V.P.) is well posed. In [KlMa] we have shown that the classical local existence result can be significantly improved for nonlinear wave equations which satisfy the null condition. Here we reexamine our results and prove that, if we restrict even further the type of nonlinearities to equations of wave maps type, we gain additional $1/2 - \varepsilon$ derivatives. This allows us to establish a result which is sharp for general equations of that type.

In [KlMa] we were considering equations of the type

$$\Box \phi^I + F^I(\phi, \partial\phi) = 0, \tag{1.1}$$

where $\Box = -\partial_t^2 + \Delta$ denotes the standard d'Alembertian in $\mathbb{R}^{n+1}$ and the nonlinear terms F have the form

$$F^I = \sum_{J,K} \Gamma^I_{JK}(\phi) B^I_{JK}(D\phi^J, D\phi^K), \tag{1.2}$$

with the B^I_{JK} having any of the null forms

$$Q_0(\phi, \psi) = \partial_\alpha \phi \cdot \partial^\alpha \psi = -\partial_t \phi \partial_t \psi + \sum_{i=1}^n \partial_i \phi \partial_i \psi \tag{1.3a}$$

$$Q_{\alpha\beta}(\phi, \psi) = \partial_\alpha \phi \partial_\beta \psi - \partial_\beta \phi \partial_\alpha \psi \qquad 0 \leqslant \alpha < \beta \leqslant n. \tag{1.3b}$$

In the particular case when only the null form Q_0 is allowed to appear in (1.2), we may say that the corresponding equations are of wave maps type. Indeed, the equations satisfied by wave maps ϕ, defined from the Minkowski space-time $\mathbb{R}^{n+1}$ to a Riemannian manifold M, take precisely that form when expressed relative to a system of local coordinates in M for which the Γ's are the corresponding Christoffel symbols.

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