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MINIMAL HEIGHTS AND POLARIZATIONS ON GROUP VARIETIES

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Let G be a commutative algebraic group, defined over a numberfield k. To a line bundle L on a compactification $\overline{G}$ of G, one can often attach a canonical height h_L on the set of algebraic points of G. Several problems of diophantine geometry, such as Lehmer's problem on the multiplicative group, or the Lang-Silverman conjecture on abelian varieties, consist in bounding from below the nonzero values of such heights. We here study how these lower bounds are affected by changes of the line bundle L itself.

Our result is most easily explained in the case when G is an abelian variety A. For a line bundle L on $\overline{G} = A$, we define h_L as the Néron-Tate height associated to L, and we write (L^g) for the g-fold intersection number of L, where g denotes the dimension of A. We then prove the existence of a positive number $c(G, k)$ such that the inequality

$$(h_L(P))^g \geqslant c(G, k)(L^g) \tag{†}$$

holds for all effective symmetric line bundles L on A/k, and all points P in $A(k)$ not contained in a proper algebraic subgroup of A. Such an hypothesis on P is clearly necessary, but even if it is not satisfied, we can still apply (†) to the identity component of the algebraic group $G(P)$, of dimension $g(P)$, generated by P in A. For a very ample line bundle L, the resulting upper bound for the degree of $G(P)$ in the corresponding projective embedding,

$$h_L(P)^{g(P)} \geqslant c'(G, k)(L^{g(P)} \cdot G(P)), \tag{$*$}$$

can be viewed as an analogue on certain one-motives of the isogeny estimates of Masser and Wüstholz [32] on abelian varieties: it yields effective bounds for the module of linear and dependence relations linking linearly dependent points on a given abelian variety. We describe this formulation in an appendix to the paper, and compare it with results of a similar kind obtained by Masser in [31].

The organization of this paper is as follows. In §1, we set the general framework of the study, and mention in passing a heuristic analogy between (†) and the isoperimetric inequalities of Alexandrov-Fenchel. In §2, we prove (†) and ($*$) in the "pure" cases where G is a torus T or an abelian variety A. The proofs rely on the construction of volume forms on various modules over the ring $\mathrm{End}(G/k)$ of

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