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ESTIMATES OF KLOOSTERMAN SUMS FOR GROUPS OF REAL RANK ONE

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1. The sum formula of Kuznetsov

1.1. Introduction. In [10] Kuznetsov gave a sum formula in which Fourier coefficients of real analytic modular forms on the upper half-plane are related to Kloosterman sums; see also [11]. This formula has been used in various ways. In [11] it is applied to the classical Kloosterman sums $S(n, m; k) = \sum_x^* e^{2\pi i(nx+m\bar{x})/k}$, where $n, m, k \in \mathbb{Z}$, $n, m \neq 0$, $k > 0$; in the sum, x runs over the integers $0 < x \leqslant k$ that are coprime to k, and satisfy $x\bar{x} \equiv 1 \bmod k$. Kuznetsov shows that

$$\sum_{1 \leqslant k \leqslant X} \frac{1}{k} S(n, m; k) = O(X^{1/6}(\log X)^{1/3}) \qquad (X \to \infty); \tag{1}$$

see Theorem 3 in [11]. This type of result is the main theme of this paper.

Kuznetsov's sum formula is concerned with automorphic forms on the group $\mathrm{SL}_2(\mathbb{R})$. It has been extended in various ways. Its extension in [17] treats automorphic forms on Lie groups of real rank one. We use it to study sums of Kloosterman sums for this class of groups.

The main structure of the sum formula is

$$\int_{\mathscr{S}} h(v)\, d\sigma(v) = \int_{i\mathbb{R}} h(v)\, d\delta(v) + \sum_{\gamma} S(\gamma)\tilde{h}(\xi_\gamma). \tag{2}$$

Here $d\sigma$ is a measure with support $\mathscr{S} \subset i[0, \infty) \cup (0, \infty)$; this measure can be described in terms of Fourier coefficients of automorphic forms for a discrete subgroup Γ of the Lie group G under consideration. The γ run over a subset of Γ, the $S(\gamma)$ are generalized Kloosterman sums, and $\xi_\gamma \in G$ is determined by γ. The measure $d\delta$ is supported on the line $\operatorname{Re} v = 0$.

Before describing the sum formula more precisely, we remark that it can be used in two directions. One way is to focus on the spectral term $\int_{\mathscr{S}} h(v)\, d\sigma(v)$. One can get information concerning the measure $d\sigma$ by taking a suitable test function and estimating the two other terms in the sum formula; see Section 2. On the other hand, for another choice of h, it may be possible to give an estimate of the spectral term and the delta term $\delta(h)$ and end up with an estimate of sums of

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