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SINGULAR SYMMETRIC MATRICES

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1. Introduction. Let $\mathscr{S}_{n,r}$ be the affine variety of $n \times n$ symmetric matrices of rank r;

$$\mathscr{S}_{n,r} = \{A \in \mathbb{C}^{n \times n} \colon A = {}^tA, \operatorname{rank}(A) = r\}.$$

We denote by $\mathscr{S}_{n,r}(\mathbb{Z})$ the integral points of $\mathscr{S}_{n,r}$. Our purpose here is to analyze the counting function

$$N(T, n, r) = \#\{A \in \mathscr{S}_{n,r}(\mathbb{Z}) \colon \|A\| < T\}.$$

$\|A\|$ denotes the standard Euclidean norm of a matrix in the space $\mathbb{R}^{n \times n}$; $\|A\|^2 = \sum_{ij} a_{ij}^2$.

There is a natural correspondence between symmetric integral matrices of rank 2 and integral quadratic forms that are decomposable over the field $\mathbb{Q}$ of rational numbers. In this context Schmidt [11] proves that

$$N(T, n, 2) = \alpha(n) T^n \log T + \begin{cases} O(T^3 \log^{1/2} T) \colon n = 3 \\ O(T^n) \colon n \geqslant 4, \end{cases} \tag{1}$$

with $\alpha(n) > 0$. In his work, Schmidt considers the slightly different norm $\|A\|_1 = \max_{ij} |a_{ij}|$. We will give an indication of his argument a little further on.

The principle results here are the following two theorems.

THEOREM 1.1. *For $n \geqslant 4$*

$$N(T, n, n-1) = \frac{n(n-1)\mathscr{V}(n)\mathscr{V}(n(n-1)/2)}{2^{2+(n-1)(n-2)/4}\zeta(n)} T^{n(n-1)/2} \log T + O(T^{n(n-1)/2} (\log T)^{1-\varepsilon(n)}), \tag{2}$$

as T tends to infinity. Here $\varepsilon(n) = (n-2)/(n-2+(n-3)(n^2+2)/3)$, which is strictly positive for $n \geqslant 3$.

Here and throughout the paper, $\mathscr{V}(k)$ is the volume of the unit ball in k dimensions, and $\zeta(k)$ is the Riemann zeta function. The notations "$f(x) \ll g(x)$" and

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