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# VOLUME-MINIMIZING CYCLES IN GRASSMANN MANIFOLDS

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In their fundamental work "Calibrated Geometries" in 1982 [HL], Reese Harvey and Blaine Lawson, discussing future global applications, posed the problem of determining the subvarieties of the real Grassmann manifolds $G_k\mathbf{R}^n$ of oriented $k$-planes through the origin in $\mathbf{R}^n$, which can be shown to be volume-minimizing in their homology classes by using as calibrations the invariant forms representing the universal Pontryagin classes.

We carry this out here in low dimensions and provide complete families of volume-minimizing representatives of all homology classes in most Grassmann manifolds $G_k\mathbf{R}^n$ for $n \leqslant 8$. The proofs "calibrate" the minimizing cycles by Kähler, quaternionic, Euler, and Pontryagin forms. The new understanding and use of the first Pontryagin form as a calibration constitutes the major technical advance of this work.

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## 1. Introduction

*1.1. Volume-minimizing cycles.* Every $d$-dimensional homology class in a compact Riemannian manifold $M$ contains a (possibly singular) $d$-dimensional "surface" of least volume. These surfaces capture much of the geometry of $M$, but are often difficult to obtain.

We begin with one of the most interesting families of volume-minimizing homology representatives in a Grassmann manifold, highlighting both our methods and point of view, and return in Section 3 of this introduction to other minimizing cycles.

Consider the 16-dimensional Grassmann manifold $G_4\mathbf{R}^8$ and its 4-dimensional real homology $H_4(G_4\mathbf{R}^8) \cong \mathbf{R}^3$. A number of 4-dimensional sub-Grassmannians

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