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RIGID PATHS OF GENERIC 2-DISTRIBUTIONS ON 3-MANIFOLDS

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Introduction. Let M be a smooth connected manifold, and E a bracket-generating (=nonholonomic) k-dimensional distribution on M (a smooth k-dimensional subbundle of TM). A smooth path $\gamma: [\alpha, \beta] \to M$ is called admissible (or E-path or horizontal) if it is tangent to E: $\dot{\gamma}(t) \in E(\gamma(t))$ for all $t \in [\alpha, \beta]$. Given two points $a, b \in M$, denote by $\Omega_E(a, b)$ the space of all E-paths $\gamma: [0, 1] \to M$ joining a to b: $\gamma(0) = a$, $\gamma(1) = b$. The space $\Omega_E(a, b)$ is not empty (by the Chow theorem) and, being endowed with a natural C^1-topology, might have singular points (called abnormal paths; several equivalent definitions can be found in [1], [5], [6], [10]) and even isolated points (called rigid paths [5]). More precisely, a path $\gamma \in \Omega_E(a, b)$ is called rigid if any C^1–close enough path of $\Omega_E(a, b)$ is a smooth reparametrization of γ. An arbitrary E-path γ defined on an interval $[\alpha, \beta]$ is called rigid if some (and then any) of its smooth reparametrization belonging to $\Omega_E(\gamma(\alpha), \gamma(\beta))$ is rigid.

Admissible, abnormal, and rigid paths can be also defined in terms of control theory. Assume for simplicity that $M = \mathbb{R}^n$. Then the distribution E is generated by k smooth independent vector fields $v_1(x), \ldots, v_k(x)$, $x \in \mathbb{R}^n$, and can be interpreted as a control system

$$\dot{x} = u_1(t)v_1(x) + \cdots + u_k(t)v_k(x), \tag{0.1}$$

where $u(t) = (u_1(t), \ldots, u_k(t))$ is an arbitrary smooth vector function (called control). The E-paths are exactly the solutions of (0.1) defined on compact intervals. The solution $x_u(t)$ corresponding to a control $u(t)$ and defined on $I = [\alpha, \beta]$ is rigid if there exists $\varepsilon > 0$ such that it is a smooth reparametrization of the solution $x_w(t)$ corresponding to a control $w(t)$, provided that the solutions join the same points ($x_u(\alpha) = x_w(\alpha)$ and $x_u(\beta) = x_w(\beta)$) and $\max_{t \in I} \|u(t) - w(t)\| < \varepsilon$.

The abnormal and rigid paths attract the attention of mathematicians working in sub-Riemannian geometry, control theory, calculus of variations, and singularity theory. The study of abnormal and rigid paths became very intensive after the discovery by R. Montgomery ([8], [10], [11]) that a rigid path might be a sub-Riemannian minimizer (the shortest admissible path joining its endpoints with respect to a given sub-Riemannian metric, i.e., a metric on E; sub-Riemannian metric allows us to measure the length of any admissible path) while it does not

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