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WHAT KINDS OF SINGULAR SURFACES CAN ADMIT CONSTANT CURVATURE?

WENXIONG CHEN AND CONGMING LI

1. Introduction. The well-known classical uniformization theorem, interpreted differential geometrically, says that every smooth Riemannian metric on a two-dimensional surface is pointwise conformal to one with constant curvature.

One would naturally ask: Is this also true for surfaces with singularities? The answer is no. Then what kinds of singular surfaces admit constant curvature? To start with, we consider surfaces with conical singularities. Roughly speaking, such a surface is a compact Riemannian surface S with a smooth metric everywhere except at finitely many points $p_1, p_2, \ldots, p_m$. Locally, near the singular point p_i, the surface is diffeomorphic to a cone with angle $\theta_i > 0$, and the metric can be written as $ds^2 = \rho(x)|x|^{-\alpha_i}|dx|^2$ in a local coordinate centered at p_i, where $\alpha_i = 2 - (\theta_i/\pi)$ and $\rho(x)$ is a smooth function. (See [8] for more details.)

These kinds of singularities appear in many situations, such as orbifolds, branched coverings, etc. They also describe the ends of complete Riemannian surfaces with finite total curvature.

In his recent paper, Troyanov [8] studied surfaces S with conical singularities. He pointed out that in the process of prescribing Gaussian curvature, according to the difficulty of the corresponding variational problem, one can use the number $\chi(S, \theta) = \chi(S) + \sum_{i=1}^{k} ((\theta_i/2\pi) - 1)$ to characterize three cases:

(i) subcritical case, $\chi(S, \theta) < \min_i\{2, \theta_i/\pi\}$,

(ii) critical case, $\chi(S, \theta) = \min_i\{2, \theta_i/\pi\}$, and

(iii) supercritical case, $\chi(S, \theta) > \min_i\{2, \theta_i/\pi\}$

where $\chi(S)$ is the Euler characteristic of S.

In the subcritical case, Troyanov showed that every conical singular metric is pointwise conformal to one with constant curvature, by using a variational method to prescribe constant curvature.

In the critical and supercritical cases, because the variational functionals lose their compactness, it becomes much more difficult to prescribe Gaussian curvature. A simple example for the critical case is a "football", the sphere with two singularities of equal order; and of the supercritical case is a "teardrop", the sphere with one singularity.

The critical and supercritical cases were considered in our previous paper [3]. We found some obstructions and obtained some sufficient conditions for a func-

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