Vol. 78, No. 2 DUKE MATHEMATICAL JOURNAL © May 1995

QUATERNIONIC TORIC VARIETIES

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1. Introduction. In this paper, we define and begin to study a new class of topological spaces analogous to real and complex toric varieties, but with the skew field of quaternions providing the underlying structure. For a convex n-dimensional polytope P and *characteristic function* λ (see Section 2 below), we define a *quaternionic toric variety* to be a certain topological quotient of $P \times (S^3)^n$. All of the identifications are prescribed by λ and occur along the proper faces of P, hence the resulting space is $4n$-dimensional. A similar topological presentation can be given for real and complex toric varieties and is the principal motivation behind our definition. To avoid confusion, we emphasize at this point that quaternionic toric varieties are *not* algebraic varieties. (In fact, exactly what one should mean by a "quaternionic variety" is not clear since the quaternions are far from commutative and polynomial maps are not well behaved.) We have adopted the name quaternionic toric variety only because of the strong and suggestive topological analogy with toric varieties.

If P is a *simple* polytope and the characteristic function satisfies certain nonsingularity conditions, the resulting quaternionic toric variety is a topological manifold whose homology Betti numbers are zero except in degrees $0, 4, 8, \ldots, 4n$. In these degrees, the Betti numbers coincide with the components of the h-vector of the polytope P; in particular, they depend only on the combinatorial type of the polytope. Moreover, the cohomology ring is a quotient of the *face ring* of P (generated in degree 4) by n independent linear relations. Both of these properties are analogous to the well-known cohomology calculations for nonsingular toric varieties and, more generally, for the toric manifolds studied by Davis and Januszkiewicz [DJ]. Nonsingularity and cohomology considerations are the content of Section 3.

Section 4 is devoted to the case $n = 2$ and, in particular, to quaternionic toric varieties which are topological 8-manifolds. The theory here is very close to the classification of compact 2-manifolds and the topological classification of simply connected 4-manifolds with effective torus actions (real and complex toric varieties being subclasses of these). These classifications present each manifold as a connected sum of manifolds of rank (middle Betti number) 1 or 2. In the 2-manifold case, rank is understood with $\mathbb{Z}/(2)$-coefficients, and the two possible summands are $\mathbb{R}P^2$ and $S^1 \times S^1$. In the 4-manifold case, the classification depends on a choice of orientations for the summands; hence there are 3 possibilities $\pm\mathbb{C}P^2$ and

Received 5 October 1993. Revision received 10 November 1994.
Partially supported by National Science Foundation grant DMS 9304580.