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EXPLICIT CONSTRUCTION OF SELF-DUAL 4-MANIFOLDS

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1. Introduction. A self-dual metric or conformal structure on a 4-manifold M is a Riemannian metric g or conformal class $[g]$ for which the Weyl conformal curvature tensor W is self-dual; M is then called a self-dual 4-manifold. Compact self-dual 4-manifolds have been extensively studied, and it is known [16] that very many compact 4-manifolds do admit families of self-dual metrics, but explicit examples of self-dual metrics that can be written down in coordinates are comparatively few. In this paper we shall provide a geometrical framework within which it is possible to construct self-dual structures by solving a linear rather than a nonlinear equation, and use it to construct some new explicit examples of compact self-dual 4-manifolds.

The two most basic compact self-dual 4-manifolds are the round metric on $\mathscr{S}^4$, which is conformally flat, and the Fubini-Study metric on $\mathbb{CP}^2$. Apart from these, the first examples of self-dual metrics on compact, simply connected manifolds were Poon's family of self-dual metrics on $2\mathbb{CP}^2$ [13]. Then LeBrun, generalizing earlier work of Gibbons and Hawking [5], found that $U(1)$-invariant self-dual metrics can be constructed from solutions to a linear equation over the hyperbolic 3-space $\mathscr{H}^3$. With this 'hyperbolic Ansatz' he wrote down a family of self-dual metrics on $n\mathbb{CP}^2$ for each n [9], which coincide with Poon's metrics when $n = 2$.

Using a similar argument, we shall study T^2-invariant self-dual metrics, and will show how to construct such metrics from solutions to a linear equation over the hyperbolic plane $\mathscr{H}^2$. Smooth actions of the torus T^2 on $n\mathbb{CP}^2$ can be classified up to equivariant diffeomorphisms [11], and for large n there are many possible actions. For every such T^2-action we shall construct a family of explicit self-dual metrics on $n\mathbb{CP}^2$, invariant under the action. One of these families for each n turns out to be a subfamily of LeBrun's metrics on $n\mathbb{CP}^2$, but the other families (for $n \geqslant 4$) are new. By the same technique we also write down explicit self-dual metrics on some other compact 4-manifolds with fundamental group $\mathbb{Z}$.

Section 1 studies 4-manifolds M equipped with a conformal structure $[g]$ and a connection ∇ preserving $[g]$. Conditions are given on the torsion and curvature of ∇ that imply $[g]$ is self-dual, but that do not require the torsion to vanish. Similar conditions are also given for $[g]$ to be conformal to an Einstein or Einstein-Weyl structure. The main inspiration for Section 1 is a formulation of the self-dual Einstein equations found by Ashtekar et al. [1].

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