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INTEGRABLE FUNCTIONAL EQUATIONS AND ALGEBRAIC GEOMETRY

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Introduction. The main goal of this paper is to show that certain functional equations can be solved using an appropriate version of the inverse spectral method. (We refer the reader to the books [1]–[5] where the basic ideas of the application of the inverse spectral method to integrable systems of differential equations are explained.) Here we shall demonstrate our method by considering in detail the functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + p(x, z). \tag{0.1}$$

Note that, in the case

$$p(x, y) = p(x - y), \qquad q(x, y) = q(x - y), \qquad r(x, y) = r(x - y), \tag{0.2}$$

where $r(z)$ is an odd function, this equation reduces to the more simple functional equation

$$\frac{q(x)q(y)}{q(x + y)} = r(x) + r(y) + p(x + y), \tag{0.3}$$

introduced by Calogero and Bruschi in connection to integrable many-body problems [6].

The usual way to solve functional equations of this type is to derive a differential equation for the involved functions assuming appropriate smoothness. This was done for equation (0.3) in [7]. (The differentiated form of the functional equation (0.3) was also used by I. Krichever [14] in his theory of action-angle variables for Calogero-Moser systems with an elliptic potential.) Here we shall solve the functional equation (0.1) without assuming smoothness or even continuity for the functions p, q, r. We only assume that these functions are Lebesgue measurable.

The equation (0.1) has a rich symmetry group. Indeed, one can transform the functions as follows:

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