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ALGEBRAIC APPROXIMATIONS OF HOLOMORPHIC MAPS FROM STEIN DOMAINS TO PROJECTIVE MANIFOLDS

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1. Introduction. The present work, which was motivated by the study of the Kobayashi pseudodistance on algebraic manifolds, proceeds from the general philosophy that analytic objects can be approximated by algebraic objects under suitable restrictions. Such questions have been extensively studied in the case of holomorphic functions of several complex variables and can be traced back to the Oka-Weil approximation theorem (see [We] and [Oka]). The main approximation result of this work (Theorem 1.1) is used to show that both the Kobayashi pseudodistance and the Kobayashi-Royden infinitesimal metric on a quasi-projective algebraic manifold Z are computable solely in terms of the closed algebraic curves in Z (Corollaries 1.3 and 1.4).

Our general goal is to show that algebraic approximation is always possible in the cases of holomorphic maps to quasi-projective manifolds (Theorems 1.1 and 4.1) and of locally free sheaves (Theorem 1.8 and Proposition 3.2). Since we deal with algebraic approximation, a central notion is that of Runge domain: by definition, an open set Ω in a Stein space Y is said to be a *Runge domain* if Ω is Stein and if the restriction map $\mathcal{O}(Y) \to \mathcal{O}(\Omega)$ has dense range. It is well known that Ω is a Runge domain in Y if and only if the holomorphic hull with respect to $\mathcal{O}(Y)$ of any compact subset $K \subset \Omega$ is contained in Ω. If Y is an affine algebraic variety, a Stein open set $\Omega \subset Y$ is Runge if and only if the polynomial functions on Y are dense in $\mathcal{O}(\Omega)$.

Our first result given below concerns approximations of holomorphic maps by (complex) Nash algebraic maps. If Y, Z are quasi-projective (irreducible, reduced) algebraic varieties, a map $f\colon \Omega \to Z$ defined on an open subset $\Omega \subset Y$ is said to be *Nash algebraic* if f is holomorphic and the graph

$$\Gamma_f := \{(y, f(y)) \in \Omega \times Z \colon y \in \Omega\}$$

is contained in an algebraic subvariety G of $Y \times Z$ of dimension equal to dim Y. If f is Nash algebraic, then the image $f(\Omega)$ is contained in an algebraic subvariety

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