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# THE CRITICAL ASYMPTOTICS OF SCALAR CURVATURES OF THE CONFORMAL COMPLETE METRICS WITH NEGATIVE CURVATURE

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**§1. Introduction and statement of the main result.** In this paper, we are concerned with the following problem **(P)**: *On a complete Riemannian manifold* $(M^n, g)$ *with* $n \geqslant 3$, *when can a smooth function* $K$ *be the scalar curvature of some conformal complete metric* $g_1$?

This question is closely related to the Yamabe problem. For the case when $M$ is compact, a lot has been done about **(P)**. Yet the answer is still incomplete. See [K], [LP] and [Sc1] for some history and [CY], [Ou1], [Ou2], and [Sc2] for some recent results. When $(M^n, g)$ is complete but noncompact, less is known. In the special case of $M^n = \mathbf{R}^n$ with standard Euclidean metric, Ni [N1] proved that $K(x)$ can be the scalar curvature of infinitely many conformal complete metrics if $|K(x)|$ decays faster than quadratic decay in a 3-dimensional subspace. This kind of decay requirement on $K$ is sharp when $K(x)$ is negative, as shown in [N1] and [L]. When $K(x)$ is positive, the situation is much more delicate (see [DN], [G1], [G2]). For the case of complete noncompact $(M^n, g)$ with negative curvature, which may be illustrated by the model case $H^n(-1)$, Aviles and McOwen [AM1] showed essentially that $K(x)$ is the scalar curvature of some complete conformal metric if it is bounded both above and below by some negative constants. (A similar result was also obtained independently by Bland and Kalka [BK] for 2-dimensional case.) Recently Jin [J1] improved this result and showed that the the behavior of $K$ at "infinity" is crucial for $K$ to be or not to be the scalar curvature of a complete conformal metric. We shall state Jin's result in detail later. Problem **(P)** for 2-dimensional manifolds may also be asked. The readers are referred to [Ah], [Av], [BK], [CTY], [M], [N2], [Ol], [Os], and [Sa], among many others, for this case.

In order to rephrase **(P)** in the formulation of partial differential equations, we write $g_1 = u^{4/(n-2)}g$ for some $u > 0$ on $M^n$. Then $u$ satisfies

$$-\gamma_n \Delta_g u + S_0 u = K u^{n^*} \quad \text{on } M^n$$

where $\gamma_n = 4(n-1)/(n-2)$, $n^* = (n+2)/(n-2)$, $\Delta_g$ is the Laplace operator with respect to the metric $g$, $S_0$ is the scalar curvature of $g$, and $K$ is the scalar curva-

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