Vol. 74, No. 2 DUKE MATHEMATICAL JOURNAL © May 1994

A FINITENESS THEOREM FOR ELLIPTIC CALABI-YAU THREEFOLDS

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0. Introduction. For the purposes of this paper, we define a *Calabi-Yau threefold* to be an algebraic threefold X over the field of complex numbers which is birationally equivalent to a threefold Y with $\mathbf{Q}$-factorial terminal singularities, $K_Y = 0$, and $\chi(\mathcal{O}_Y) = h^1(Y, \mathcal{O}_Y) = h^2(Y, \mathcal{O}_Y) = 0$. In this case, we call Y a minimal Calabi-Yau threefold.

Calabi-Yau threefolds can be thought of as a generalization of K3 surfaces. If one considers possibly nonalgebraic K3 surfaces, they are all Kähler, and one obtains an irreducible 20-dimensional moduli space. Thus in particular, all K3 surfaces are homeomorphic. If one restricts one's attention to algebraic K3 surfaces, the situation becomes much more complicated: one obtains a countable number of 19-dimensional components in the 20-dimensional space of Kähler K3s. In the case of Calabi-Yau threefolds, however, any deformation of an algebraic Calabi-Yau is algebraic, so it makes sense to restrict one's attention to algebraic Calabi-Yaus.

Given this, one could ask whether there are a finite number of topological types of algebraic Calabi-Yau threefolds. (This is known not to be true if one allows non-Kähler Calabi-Yau threefolds.) A stronger question to ask would be whether there are a finite number of families of algebraic minimal Calabi-Yau threefolds. Up to birational equivalence, we answer this stronger question for those Calabi-Yaus which possess an elliptic fibration.

Our main theorem is the following.

THEOREM 0.1. *There exists a finite number of triples $(\mathcal{X}_i, \mathcal{S}_i, \mathcal{T}_i)$ of quasi-projective varieties with maps*

$$\begin{array}{ccc} \mathcal{X}_i & & \\ \downarrow f_i & \searrow \pi_i & \\ \mathcal{S}_i & \xrightarrow{g_i} & \mathcal{T}_i \end{array}$$

where π_i is smooth and proper with each fibre a Calabi-Yau threefold, f_i proper with generic fibre an elliptic curve, and g_i smooth and proper with each fibre a rational surface, such that for any elliptic fibration $X \to S$ with X Calabi-Yau and S rational there exists a $t \in \mathcal{T}_i$ for some i such that there are birational maps $X \dashrightarrow (\mathcal{X}_i)_t$,

Received 21 May 1993. Revision received 5 August 1993.
Research at MSRI supported in part by NSF grant DMS 9022140.