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THE CENTRAL LIMIT THEOREM FOR THE GEODESIC FLOW ON NONCOMPACT MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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The aim of this paper is to present a new method for studying quantities of the form $\int_0^t \varphi(\theta_s \xi)\, ds$, θ_s being the geodesic flow of a manifold of constant negative curvature and finite volume X and φ being a function on $T_1 X$. The basic idea is to notice that the integral of φ along the geodesic coincides with the integral of the 1-form closed along the stable leaf defined by the geodesic and its horocycles at $+\infty$, and then to change the integration path, replacing the geodesic by a diffusion path on this leaf, following the geodesic towards $-\infty$. We can apply it to extend the Ratner-Sinaï central limit theorem [R], [S] to noncompact manifolds when φ is smooth.

A new formula for the variance follows from the proof, which does not rely on any coding argument. This method is likely to be applicable to further problems, and its scope should be extendable to more general situations. It is an elaboration of the method used in the note [L] on the windings of geodesics (which treated also the windings around the cusps, yielding Cauchy distributions), which was itself a development of [GL]. In the proof, an important role is played by a spectral gap argument yielding a potential operator for the geodesic diffusion. The idea of using a Sobolev space instead of L^2 stems from an analogy with [GH] where the spectral gap occurs in the space of Lipschitz functions.

We treat in detail the two-dimensional case, which here contains all the ideas and is more often presented in the literature. Indications for the d-dimensional case are given in the last paragraph.

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1. Framework and statement of the result. Let Γ be a discrete subgroup of $SL^2(\mathbb{R})$ containing $-I$, without elliptic elements. $SL^2(\mathbb{R})$ acts by left multiplication on the hyperbolic plane $H = SL^2(\mathbb{R})/SO(2)$, and $\Gamma\backslash H$ is the Riemann surface attached to Γ. We assume it has finite volume. $\Gamma\backslash SL^2(\mathbb{R})$, denoted by F, can be identified with the unitary tangent bundle of $\Gamma\backslash H$ in such a way that the geodesic flow maps in time t $\xi \in F$ into $\xi\theta_t$ with $\theta_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. We denote by $m(d\xi)$ the normalized Liouville measure on F, which is induced by the Haar measure on $SL^2(\mathbb{R})$. $SL^2(\mathbb{R})$ acts by right multiplication on F. This action preserves m, and the geodesic flow is known to be ergodic.

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