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LIMITS OF TANGENT SPACES TO FIBRES AND THE w_f CONDITION

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Let $f: U \to \mathbf{C}$ be a holomorphic function defined on an open neighbourhood U of the origin in $\mathbf{C}^{n+1}$ such that $f(0) = 0$ and 0 is the only (possible) critical value of f. The main purpose of this paper is to prove the following theorem.

THEOREM. *Let $\mathscr{S} = \{S_i\}$ be a Whitney stratification of $X = f^{-1}(0)$. Then, for each stratum S of $\mathscr{S}$ the pair $(U\backslash X, S)$ satisfies the w_f condition.*

(We say that $(U\backslash X, S)$ satisfies w_f if for every $x_0 \in S$ there is a neighbourhood U_{x_0} in U and a constant C such that for every $x \in U_{x_0}\backslash X$,

$$\delta(T_{x_0}S, T_x f^{-1}(f(x))) \leqslant C\|x - x_0\|,$$

where the left-hand side denotes the angle between $T_{x_0}S$ and the tangent space to the level of f at x; see [HMS] for the details.)

The theorem is proven in Section 2. Section 1 contains the necessary preparation for the proof, that is, a local analysis on the (projectivized) relative conormal space C_f of f, or rather its blow-up $E_Y C_f$, where Y is a nonsingular subset of U (see the diagram (1.1) below), and a study of the exceptional divisor $\mathscr{E}$ in $E_Y C_f$, which can be interpreted as the space of the limits of secants and tangent hyperplanes to the levels of f. This construction appeared and was developed before in several places both for the absolute conormal spaces (see [BS], [HM2], [LT], [T2]) and for the relative ones [HMS]. In [LT, Corollaire 2.1.3] the authors give a beautiful characterization of such space of limits in the absolute case as a finite union of dual correspondences. (See a remark on terminology below.) As a by-product of our method we give in Proposition 1 below a similar description of such spaces in our case, which, however, does not characterize them completely.

Assume that the set of singular points ΣX of X is itself nonsingular, and consider the pair $(X\backslash\Sigma X, \Sigma X)$ as a stratification of X. In this case the result of the theorem is well known since then both the Whitney conditions and the w_f condition are equivalent to the μ^*-constant condition [T1], [BS], and [HMS]. The author's original idea for the proof of the theorem was to generalize this argument using polar multiplicities and Lê numbers ([M]) of the generic plane sections (note that both of these families generalize μ^* numbers) and use Proposition 2.9 of [M]. An attempt to make this idea precise led to the study of the geometry of conormal

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